SRAFFA'S REDUCTION TO DATED QUANTITIES OF LABOUR — A NOTE

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Introduction

In Sraffa's well-known work,[1] the method of explaining the (exchange) value of commodities as the present value of the total (direct and indirect) labour applied in their production was explained. The purpose of this procedure was the explicit demonstration of the impossibility of determining the quantity of capital as a measure independent of relative prices and distribution. Here the crucial role is designated to the variations of values of corresponding row members, depending on the profit rate and the time interval and their reaching the maximum values for different pairs of these variables. Analytically, it is possible to identify the connection between the maximum value of any member of the row and the corresponding rate op profit, and vice-versa.

In this paper, consideration is given to the problem of choosing that row member which, for a certain value of the rate of profit, has a value greater than any other member of the row.

The procedure of reduction to dated quantities of labour

Under conditions of (1) long-run stationary equilibrium, (2) complete consumption of capital means with one turnover, (3) paying wages at the end of the turnover period and therefore *eliminating* them from the fund of advanced capital, for any "industry" the following holds:

$$(1) \qquad C_o (1+r) + L_o w = Qp$$

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1) P. Sraffa, "Production of Commodities by Means of Commodities", Chapter VI, Cambridge University Press, 1976.

where:

C_o — the value of the means of production;
L_o — the quantity of labour directly applied in $t = 0$;
Q_o — the quantity of the produced commodity;
r — rate of profit;
w — wage rate;
p — the price of the commodity.

Successively expressing the means of production as the value of production of a previous "date", after "k" steps we obtain

$$C_{-1}(1+r) + L_{-1}w = C_o$$

$$C_{-2}(1+r) + L_{-2}w = C_{-1}$$

(2) ....

....

$$C_{-k}(1+r) + L_{-k}w = C_{-k+1}$$

where (C_{-s}) and (L_{-s}) are capital and labour applied in the production "s" years earlier.[2]

Now equation (1) may be written in its reduced form:

(3) $$L_o w + L_{-1}w(1+r) + L_{-2}w(1+r)^2 + \dots$$
$$\dots + L_{-k}w(1+r)^k + C_{-k}(1+r)^{k+1} = Qp$$

It is obvious that after a *finite* many number of steps the value of the commodity can not be expressed solely as the value of applied labour (because of the term $C_{-k}(1+r)^{k+1}$), excluding the hypothetical case that in one of the steps the commodity is produced only by applying labour, that is when $C_{-k} = 0$ and when $wL_{-1} = C_{-(k-1)}$. However, if the process of "converting" capital into labour continues indefinitely, the term C_{-k} tends to zero, and the left side of equation (3) becomes a row[3] whose every member represents the present value of labour applied at the appropriate step:

(4) $$L_o w + L_{-1}w(1+r) + \dots + L_{-n}w(1+r)^n + \dots \qquad (= Qp)$$

The value of the sum of the row depends solely upon factor prices "r" i "w" and the series $L_o, L_{-1}, \dots, L_{-n}, \dots$, which we shall call the "history" of labour.

When $r = 0$, that is when capital as a factor of production has no price, the value of commodity (Qp) is identical to the value of the sum of total labour applied in its production:[4]

2) The negative sign of the index shows that the process toward the past is under consideration.

3) This row converges to Qp by definition.

4) Conversely, for $w = 0$, when labour has no price as a factor of production and ceases to be the object of economic calculation.

(5) $$(L_o + L_{-1} + L_{-2} + L_{-n} + \dots)\, w = Qp$$

If the prices and wages are expressed in standard commodity[5] the following relationship holds between the wage (as a proportion of the standard commodity) and the rate of profit:

$$w = 1 - \frac{r}{R}$$

where R is the maximum rate of profit. Then the general member of the row (denoted as $\Theta_n(r)$)

(6) $$\Theta_n(r) = L_{-n}\left(1 - \frac{r}{R}\right)(1+r)^n$$

is the function exclusively of "n" and "r" (where it is important to note L_{-n} depends only on "n".

Adjusting the results

For any randomly selected "vintage" (that is, for every fixed n), the dependence of Θ_n on r can be represented by the curve in the coordinate system (Θ, r). From the condition $0 \leq w \leq 1$, which reflects the economic requirement that wages can not exceed the net standard product, it follows that $\Theta_n(r)$ is defined only for $0 \leq r \leq R$. For economic reasons, the inequality $\Theta_n(r) \geq 0$ always holds and immediately follows L_{-n} 0. The term $\Theta_n(r)$ has the zero value only if $r = R$ and/or if $L_{-n} \geqq 0$.

The *values* of maxima of curves $\Theta_n(r)$ *depend* on the concrete "history" of labour $L_o, L_{-1}, \dots, L_{-n}, \dots$. However, the *locations* of maxima (the values of rates profit for which the curves reach their maxima) do *not depend* on the "history" of labour. This is due to the fact that the changing history of labour corresponds to the introduction of new constant (independent of r) multipliers L_{-n} for every curve $\Theta_n(r)$ which somehow alters only the shape of the curves without changing the location of the maxima.

It appears (See Appendix) that the dependence of the location of maxima r_{max} on n is determined by the relation:

(7) $$r_{max} = \frac{nR - 1}{n + 1}$$

Let us pose the following problem: for the fixed r, which member of the row

5) P. Sraffa, *op. cit.*, pp. 20—24.

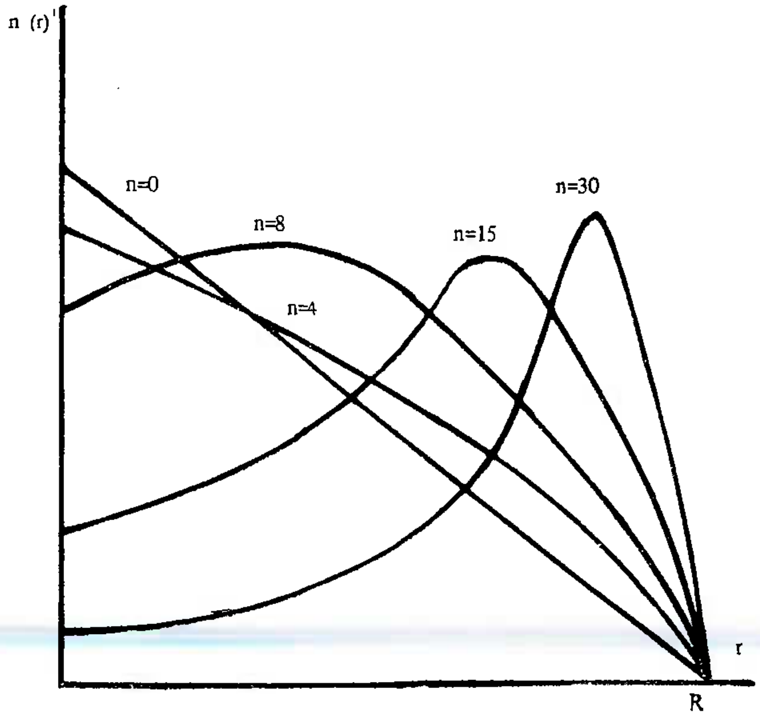

Fig. 1. — Variation in value of different members of the row for different periods and different values of the rate of profit.

$$(8)\qquad L_0\left(1-\frac{r}{R}\right)+L_{-1}\left(1-\frac{r}{R}\right)(1+r)+\ldots+L_{-n}\left(1-\frac{r}{R}\right)(1+r)^n+\ldots$$

has a value greater than any other member of the row? The solution depends, obviously, on the history of labour.

Let us denote this unknown member by index n_{max}, and let us introduce two additional assumptions:

(a) that the curves $\Theta(n, r)$ are distributed "densely" in the coordinate system (Θ, r) (n is treated as a continuous parameter);

(ii) the assumption that there exists such a history of labour for which the maxima of all curves from the family lie on the line $\Theta = 1$ (or $C =$ const. in the general case).

From (ii) we can obtain an explicit dependence L_{-n} on n. That is, the condition

$$(Vn)\quad \max \Theta\, n, r = 1$$

is equivalent to the condition

$$(Vn)\quad L_{-n\ r}^{\ max}\left(1-\frac{r}{R}\right)(1+r)^n = 1$$

from which we directly obtain

$$(9)\qquad L_{-n} = \frac{1}{\left(1-\frac{r_{max}}{R}\right)(1+r_{max})^n} = \frac{R}{(R+1)}\,\frac{(n+1)}{(R+1)^n}\left(1+\frac{1}{n}\right)^n$$

because $r_{max}^{(n)} = \dfrac{nR-1}{n+1}$ (see (7))

Now, when the dependence L_{-n} on n is known, the determination of n_{max} is reduced to solving the equation:

$$\frac{\partial\,\Theta^*(n, r)}{\partial n} = 0$$

where $\Theta^*(n, r)$ is the family $\Theta(n, r)$ for the special choice of history of labour (9).

It is shown (see Appendix) that

$$(10)\qquad n_{max} = \frac{1+r}{R-r}$$ [6]

It is easy to show the inversion of (7) and (10):

$$n_{max}(r_{max}(n)) = \frac{1+\dfrac{nR-1}{n+1}}{R-\dfrac{nR-1}{n+1}} = \frac{n(R+1)}{R+1} = n$$

and

$$r_{max}(n_{max}(r)) = \frac{\left(\dfrac{1+r}{R-r}\right)R-1}{\dfrac{1+r}{R-r}+1} = \frac{r(R+1)}{R+1} = r$$

6) Let us note that Sraffa obtains relation (10) by direct inversion of (7). See, Sraffa, ibid., p. 37.

If we abandon the requirement contained in the assumption (ii), that is if we withdraw the special choice of the history of labour (9), we will have in the general case

(11) $(\forall n) \max_r \Theta(n, r) = f(n)$

where f (n) is some function of n.

If the function f (n) is known, then from (11) we directly obtain:

$$L_{-n} = \frac{f(n)}{\left(1 - \frac{r_{max}}{R}\right)(1 + r_{max})^n} = \frac{R}{R+1}\,\frac{n+1}{(R+1)^n}\left(1 + \frac{1}{n}\right)^n f(n)$$

that is

$\Theta\, n, r = \Theta^*(n, r)\, f(n).$

The equation $\frac{\partial \Theta(n, r)}{\partial n} = 0$ is equivalent (with respect to n) to the following equation:

$$\left(\frac{1+r}{1+R}\right)\left(1 + \frac{1}{n}\right) = e^{-s_f(n)}$$

where $s_f(n) = \dot{f}/f$ (see appendix III).

Obviously,

$$\left(\frac{1+r}{1+R}\right)\left(1 + \frac{1}{n}\right) = 1$$

if and only if $s_f(n) = 0$, that is if and only if f = const.

In other words,

$$n_{max} = \frac{1+r}{R-r}$$

if and only if f = const.

If we withdraw the requirement contained in the assumption (i), that is if we assume that n takes only descreete values, the alternative procedure must be undertaken. For the purpose of determining n_{max} for any assumed dependance of L_{-n} on n (that is for any "history" of labour), let us observe the intersection of curves which have neighbouring *locations* of maxima.

The locations of these intersections (which we shall denote as $r_o(n)$) are obtained by solving the equation:

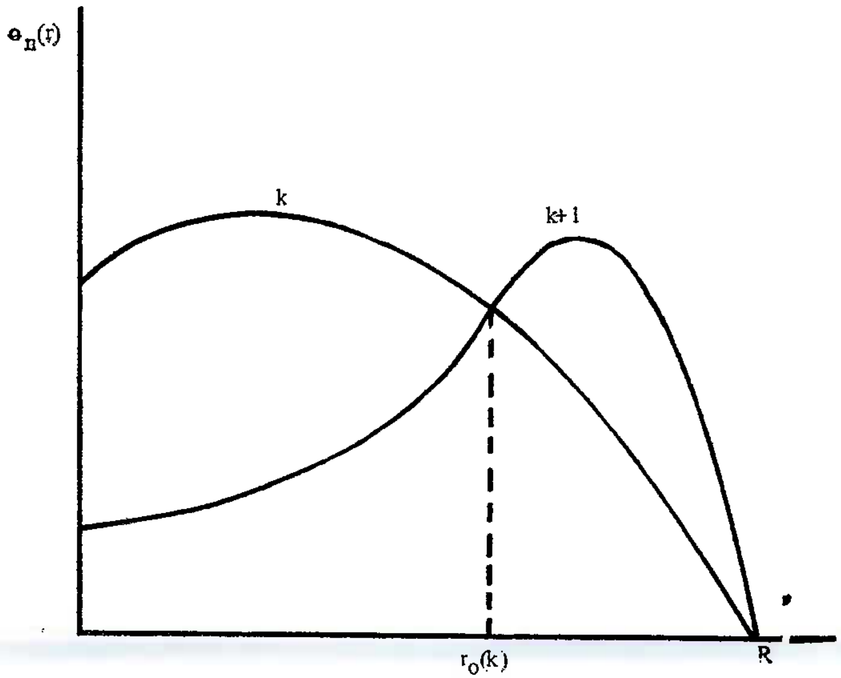

Fig. 2. Curves with neighbouring location of maxima

(12) $\Theta_n(r_o(n)) = \Theta_{n+1}(r_o(n)),$ under constraint $R > r_o(n) \geqslant 0$

which is equivalent to:

(13) $$L_{-n}\left(1 - \frac{r_o(n)}{R}\right)(1 + r_o(n))^n = L_{-(n+1)}\left(1 - \frac{r_o(n)}{R}\right)(1 + r_o(n))^{n+1}$$

from which, dividing both sides of the equation with the common member $\left(1 - \frac{r_o(n)}{R}\right)$ $(1 + r_o(n))^n$ and solving for $r_o(n)$, it follows:

(14) $$r_o(n) = \frac{L_{-n}}{L_{-(n+1)}} - 1, \quad \text{if } L_{-(n+1)} \neq 0.$$

From the condition $r_o(n) = 0$ and (14) it follows:

$L_{-n} \geqq L_{-(n+1)}$

and from $r_o(n) < R$ and (14)

$$L_{-(n+1)}(R+1) > L_{-n} \quad .$$

These inequalities define the class of the history of labour for which the intersection always exists.

Therefore, for such a known dependence of L_{-n} on n, the dependence r_{max} on n is also known.

r (n) generates a series of points on the r-axis. These points divide the interval [0, R] on a series of subintervals $[0, r_o(1)]$, $[r_o(1), r_o(2)]$, ... , $[r_o((k-1), r_o(k)]$ ..... .

The procedure for determining n_{max} for a fixed r is based on recognising the subinterval in which r is contained.

Let us assume that r is contained *within* the interval $[r_o(k-1), r_o(k)]$. Then, $n_{max} = k$. If, however, r *coincides* with one of the limits of the interval, for instance with $r_o(k-1)$, then n_{max} has two values, k and (k — 1).

Conclusions

Regarding the row expressing the exchange value of a commodity as the present value of total labour applied in its production, the problem of choosing that member of the row which has a value greater than any other member (for a certain value of the rate of profit) is considered. In the case of a particularly-defined history of labour, which equalizes the value of every member of the row to some arbitrary set constant, the results obtained are identical with those of Sraffa. In this paper, the distinction between continous and discreet cases was made as well.

APPENDIX

(I) r_{max} is the solution of the algebraic equation

$$(A1) \qquad \frac{\partial}{\partial r}\Theta(n,r)\Big|_{r=r_{max}} = 0$$

Since L_{-n} does not depend on r, (A1) is equivalent to

$$\frac{\partial}{\partial r}\left(1-\frac{r_{max}}{R}\right)(1+r_{max})^n = 0$$

that is to

$$-\frac{1}{R}(1+r_{max})^n + \left(1-\frac{r_{max}}{R}\right)n(1+r_{max})^{n-1} = 0 \quad .$$

From this we directly obtain

$$r_{max} = \frac{nR-1}{n+1}$$

(II) n_{max} is the solution of the algebraic equation

$$(A2) \qquad \frac{\partial}{\partial n}\Theta^*(n,r)\Big|_{n=n_{max}} = 0$$

where

$$\Theta^*(n,r) = \frac{R}{R+1}(n+1)\left(\frac{n+1}{n}\right)^n \frac{1}{(R+1)^n}\left(1-\frac{r}{R}\right)(1+r)^n$$

(for the dependence L_{-n} on n the relation (12) is used).

(A2) is equivalent to

$$(n_{max}+1)\left[\ln(1+r) - \ln(1+R) + \ln\left(1+\frac{1}{n_{max}}\right)\right] = 0$$

that is to

$$\left(\frac{1+r}{1+R}\right)\left(1+\frac{1}{n_{max}}\right) = 1 \quad .$$

From this we directly obtain

$$n_{max} = \frac{1+r}{R-r} \quad .$$

(III) n_{max} is the solution of the algebraic equation

$$(A3) \qquad \frac{\partial}{\partial n}\Theta(n,r)\Big|_{n=n_{max}} = 0$$

where $\Theta(n,r) = \Theta^*(n,r)\, f(n)$

(3) (A3) is equivalent to

$$\frac{\partial\Theta}{\partial n} f(n) + \Theta\frac{\partial f}{\partial n} = 0$$

that is to

$$\Theta\left[f(n)\ln\left[\left(\frac{1+r}{1+R}\right)\left(1+\frac{1}{n}\right)\right] + \frac{\partial f}{\partial n}\right] = 0$$

From the last equation it follows

$$\ln\left[\left(\frac{1+r}{1+R}\right)\left(1+\frac{1}{n}\right)\right] = -\frac{\frac{\partial f}{\partial n}}{f}$$

that is

$$\left(\frac{1+r}{1+R}\right)\left(1+\frac{1}{n}\right)=e^{-s_f(n)}$$

where $s_f(n)$ is the rate of growth of the function f(n).

SRAFFINO SVOĐENJE PROMETNE VREDNOSTI NA UKUPNU KOLIČINU RADA: KOMENTAR

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Rezime

Kod Sraffinog metoda izražavanja prometne vrednosti roba (kao sadašnje vrednosti sume ukupnih direktnih i indirektnih radova koji su uloženi u njihovu proizvodnju) ključnu ulogu imaju varijacije vrednosti odgovarajućih članova reda u zavisnosti od veličine profitne stope i vremenskog intervala, a u cilju eksplicitnog dokaza da se veličina kapitala ne može odrediti nezavisno od odnosa raspodele. U ovom radu razmatra se problem nalaženja onog člana reda koji za određenu vrednost profitne stope ima vrednost veću od bilo kog drugog člana tog reda. U radu je pokazano da se za slučaj specijalno izabrane »istorije« radova, koja član normira na proizvoljnu konstantu dobijaju rezultati identični onima koje je dobio Sraffa. Takođe je dat algoritam problema za proizvoljno izabranu »istoriju« radova. U analizi je razlikovan diskretan i kontinuelan slučaj.

JEDNA SPECIFIČNA PRIMENA DINAMIČKOG PROGRAMIRANJA: APROKSIMACIJA VREMENSKIH SERIJA ODSEČCIMA VIŠE PRAVIH LINIJA

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1. UVOD

Pri praćenju ponašanja ekonomskih pojava često se postavlja zadatak nalaženja »najbolje« aproksimacije vremenskih serija zadatim tipovima funkcija. Zadatak se može rešavati na razne načine, koristeći raznovrsne gradijentne postupke i tehnike numeričkog pretraživanja. U ovom radu je opisan postupak nalaženja »najbolje« aproksimacije vremenskih serija odsečcima više pravih linija. Ovakva aproksimacija poseduje sve dobre osobine linearnih aproksimacija tj. omogućava dalju linearnu analizu posmatranih pojava, pa je potreba za njom česta. Zatim, činjenica da vremensku seriju podataka, umesto jednom pravom, aproksimiramo pomoću odsečaka više pravih, govori da će ovakva aproksimacija biti kvalitetnija, naročito kada je u pitanju duža vremenska serija.

Uočavaju se dve vrste ovih zadataka. U prvoj vrsti polazni podaci su predstavljeni serijom originalnih podataka datih u obliku niza uređenih parova brojeva. U drugoj vrsti zadataka vremenska serija je već preliminarno analizirana i opisana nekom analitičkom nelinearnom krivom koja je dobijena aproksimacijom originalnih podataka, pa u sledećem aproksimativnom koraku u analizi, krivu treba »zameniti« odsečcima više pravih. Na primer, pojave koje pokazuju tendenciju stalnog rasta (kretanje društvenog proizvoda privrede u celini ili pojedinih privrednih grana, broj zaposlenih, učešće industrijske proizvodnje u ukupnom društvenom proizvodu) obično se najpre predstavljaju eksponencijalnim krivim, a zatim, u cilju dalje analize, aproksimiraju odsečcima više pravih linija.

U obe ove vrste zadataka reč »najbolja« aproksimacija odsečcima više pravih linija se odnosi na unapred izabranu meru kvaliteta aproksimacije. U praktičnim zadacima mera je najčešće suma kvadrata razlike aproksimacije i stvarne vrednosti. Međutim, mera kvaliteta mo-

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