THE LOCATION OF THE OPTIMAL ROUTE ON A RECTANGULAR SURFACE*)

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The surface on which it is required to solve the problem of the optimal route has, as seen in fig. 1. the shape of a rectangle leaning uniformly toward the base-line a so that the isohipses are parallel to the base-line and perpendicular to the side-line b of the rectangle. We place a perpendicular Cartesian coordinate system $x0y$ on the surface; the absciss and ordinate axes coincide with the base-line and the left side-line of the rectangle, respectively. Each surface of the area is, in a chosen time unit, an origin of the substratum to bet transported; we presume that the density of the substratum is uniform over the entire rectangle and equal to g. We trace a single route connecting the right side-line and the prescribed point $E(0.q)$ on the left side-line of the rectangle; the location of this route is determined according to the optimum criteria which requires the minimization of the total transportation costs.

With the model under consideration only the following types of expenditures are taken into account:

1. Transportation costs along natural routes from the origins to the route under consideration; these routes flow along the fall lines and are therefore parallel to the side-lines of the rectangle. We presume that these costs are linearly proportionate to the quantity of substratum m, the length of the route s and the transportation costs p. Because of the inclination of the surface we distinguish the costs p_1 of transport downward from the costs p_2 of transport upward to the route, where $p_1 < p_2$.

2. Transportation costs along the route itself. We presume that these costs are lineally proportionate to the quantity of the transported substratum m, the length s of the route and the transportation costs p_3.

3. Costs involving the means of transportation. Among these costs only the ones depending upon the length of the route are taken into account. We presume that these costs are linearly proportionate to the

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lenght s of the route and the costs p_4 in a length unit of the route, reduced to a unit of time.

In the numerical example included, the parameters assume the following values:

$$a = 2000\, m$$
$$b = 1200\, m$$
$$g = 0001\, t/m^2,$$

while the costs expressed in monetary units are:

$$p_1 = 20$$
$$p_2 = 60$$
$$p_3 = 1$$
$$p_4 = 8000$$

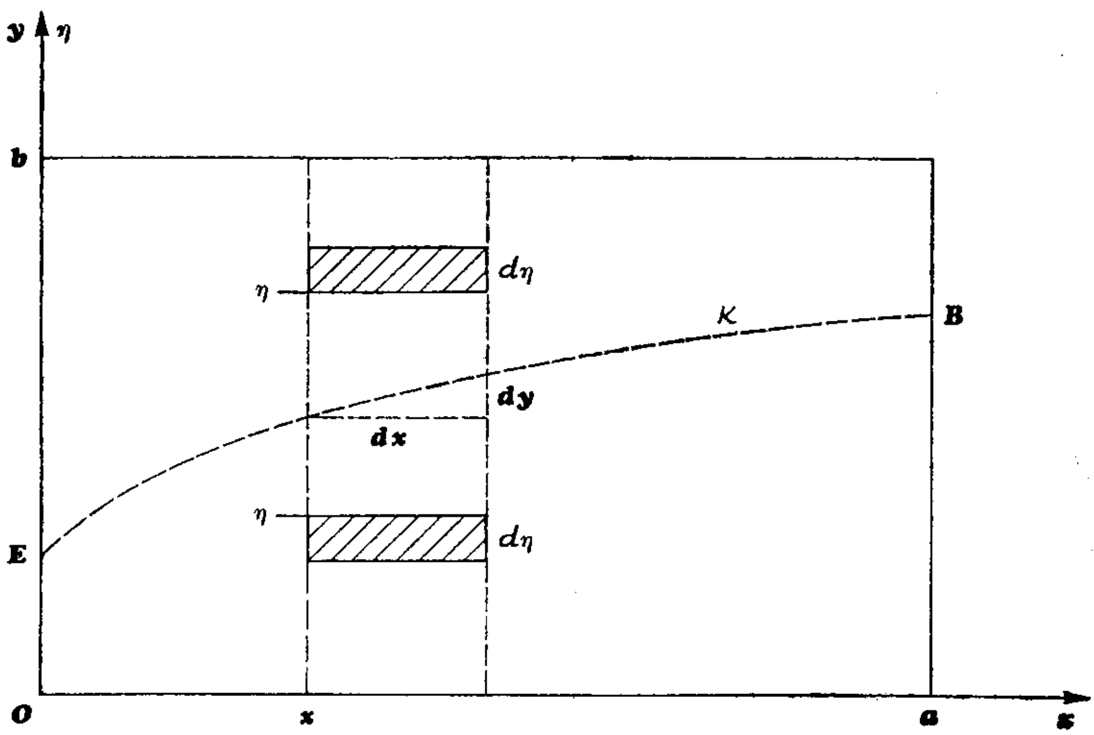

Fig. 1. A route, terminating at a fixed point E.

Suppose the function $y = y(x)$ defines the trace K of an artificial route; the function satisfies the boundary conditions:

$$x = 0 \; q \; y = q$$

The artificial route is designed so as to minimize the compound transportation costs it induces. To meet this purpose let us calculate the compound costs of transportation of the substratum from the entire space to the termination point E(0, q) of the route; these costs are quadruple:

1. The costs of transport of the substratum along a natural route downward to the fare; these costs equal:

$$T_1 = \int_0^a dx \int_y^b gp_1 \, (\eta - y) \, d\eta = \frac{1}{2} \, gp_1 \int_0^a (b - y)^2 \, dx$$

2. The costs of transportation of the substratum along a natural route upward to the fare; these costs equal:

$$T_2 = \int_0^a dx \int_0^y gp_2(y - \eta) d\eta = \frac{1}{2} \, gp_2 \int_0^a y^2 \, dx$$

3. Transportation costs along the fare; these costs equal:

$$T_3 = bgp_3 \int_0^a dx \int_0^x \sqrt{1 + (dy/dx)^2} \, dx$$

4. The costs of establishment and maintenance of an artifical route; these equal:

$$T_4 = p_4 \int_0^a \sqrt{1 + (dy/dx)^2} \, dx$$

By adding all these costs we obtain the compound costs of transportation of the substratum from the entire space:

$$T(y(x)) = \int_0^a \Big[\frac{1}{2} \, g \, p_1 \, (b - y)^2 + \frac{1}{2} \, g \, p_2 \, y^2 + \\ + b \, g \, p_3 \int_0^x \sqrt{1 + (y')^2} \, dx + \\ + p_4 \sqrt{1 + (y')^2} \Big] \, dx$$

The following transformation of the third term:

$$\int_0^a b \, g \, p_3 \left(\int_0^x \sqrt{1 + (y')} \, dx \right) dx =$$

$$= b \, g \, p_3 \left[x \int_0^x \sqrt{1 + (y')^2} \, dx \Big|_0^a - \int_0^a x \sqrt{1 + (y')^2} \, dx \right] =$$

$$= b \, g \, p^3 \int_0^a (a - x) \sqrt{1 + (y')^2} \, dx$$

yields:

$$T\,(y\,(x)) = \int_0^a \left[\frac{1}{2}\, g\, p_1\, (b - y)^2 + \frac{1}{2}\, g\, p_2\, y^2 + (b\, g\, p_3\, (a - x) + p_4) \sqrt{1 + (y')^2} \right] dx \quad (1)$$

These costs depend upon the funotion y(x). This function is to be determined so as to minimize the transportation costs T(y(x). Consequently a problem of the calculus of variations is to be solved. In this case the functional, the minimum of which is to be determined, is of the type:

$$J\,(y) = \int_0^a f\,(x,\, y,\, y')\, d\,x$$

Since the functional is of this type that enables the application of the Euler's differential equation of the calculus of variations:

$$\frac{\partial f}{\partial y} = \frac{d}{d\,x} \left(\frac{\partial f}{\partial y'} \right) \quad (2)$$

The derivatives in the Euler's differential equation equal:

$$\frac{\partial f}{\partial y} = g\,(p_1 + p_2)\, y - b\, p_1$$

$$\frac{\partial f}{\partial y'} = (b\, g\, p_3\, (a - x) + p_4)\, \frac{y'}{\sqrt{1 + (y')^2}}$$

$$\frac{d}{d\,x}\, \frac{\partial f}{\partial y'} = -\, b\, g\, p_3\, \frac{y'}{\sqrt{1 + (y')^2}} + (b\, g\, p_3\, (a - x) + p_4)\, \frac{y''}{\sqrt{(1 + (y')^2)^3}}$$

Substituting these expressions into (2) and arranging the terms we obtain the final form of the Euler's differential equation of the variational problem under consideration:

$$g\,(p_1 + p_2)\, y + b\, g\, p_3\, \frac{y'}{\sqrt{1 + (y')^2}} - (b\, g\, p_3\, (a - x) + p_4)\, \frac{y''}{\sqrt{(1 + (y')^2)^3}} = b\, g\, p_1 \quad (3)$$

It is not likely that we would obtain the general solution of this second-order differential equation by means of successive integrations. Therefore in our further deliberations we shall consider only the more important particluar solutions of the locational problem. These particular solutions satisfy the chosen boundary conditions and can be obtained either directly from the differential equation or computed numerically by means of a digital computer.[4]) The general solution of the dif-

[4]) The program for numerical calculation of a particular extremal dependent upon given boundary points B & E is deposited in the program library of the Institute of Transportation.

ferential equation is represented by a two-parameter family of extremals; each of them originates from an initial point B on the right side-line and terminates in a point E on the left side-line of the rectangular space.

First let us answer the question of the existence of an extremal parallel to the base-line of the rectangle. Since such an extremal is obviously a straight line parallel to the abscissa, its first and second derivative assume the value 0 at all points. Substituting these conditions into (3), we obtain the equation:

$$g(p_1 + p_2) = bgp_1,$$

which yields:

$$y = \frac{bp_1}{p_1 + p_2}$$

We denote the fraction $(bp_1)\ (p_1 + p_2)$ by the letter s. Thus we have confirmed the existence of an extremal parallel to the base-line of the rectangle; it originates in the point $B_s\ (a, s)$, passes at the height s and terminates in $E_s\ (0, s)$. Given numerical data[5]), this route is represented on fig. 2., whence s = 300 m.

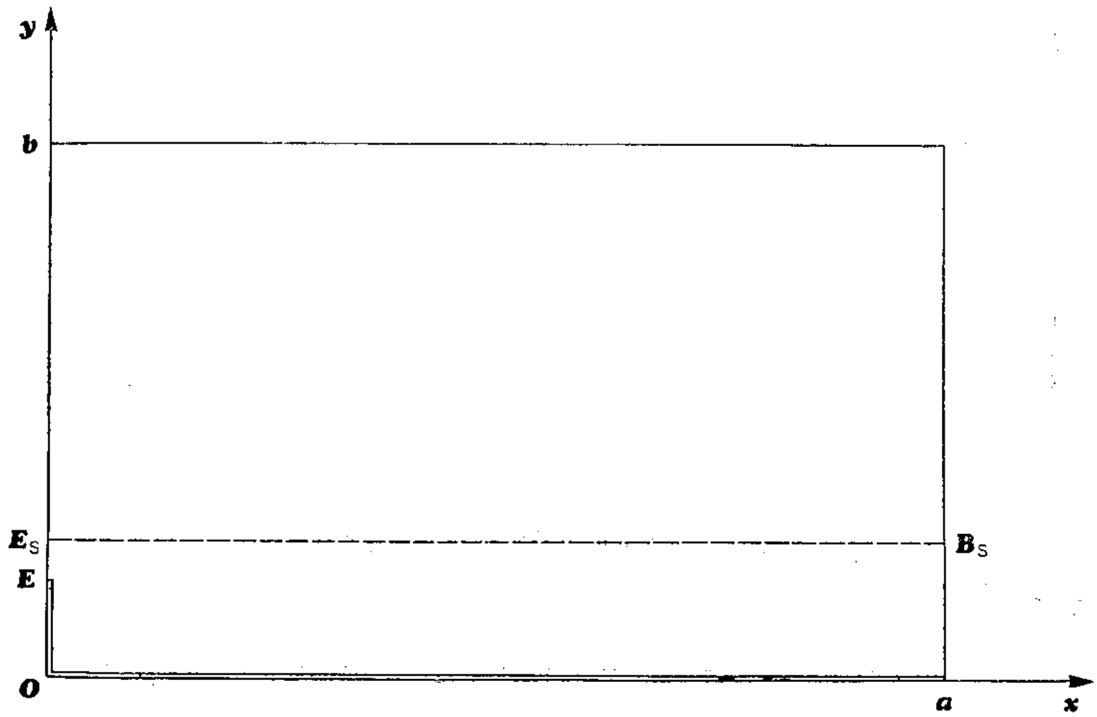

Fig. 2. A total extremal

[5]) As previously stated the data with the problem under consideration are: a = 2 000 m, b = 1 200 m, g = 0,001 t/m²; p_1 = 20, p_2 = 60, p_3 = 1 and p_4 = 8 000 monetary units.

The advantage of this route over all other possible routes is that the costs of transportation of the substratum to the route are minimized along is entire length and each particular section as well; therefore it is referred to as the *totas extremal.* This route connects rectilinearly the point B_s and E_s and partitions, as seen on fig. 3., the rectangular space into two parts; the lower part under the route and the upper part over the route.

The total extermal which satisfies the equation $y = s$ is of exceeding importance in the locational problem under consideration. Since the costs of transportation of the substratum to this route are minized along each of its sections, every other extremal, boundary and initial condition taken into account, tends to approach the latter as close as possible. According to the properties of extremals supplied by the theory of the calculus of variations, two extremals terminating in a given point E do not intersect; therefore only one extremal terminating in a given point E passes through each point of the rectangle. The same applies to the extremals originating from a given point B. Hereafter we shall consider extremals with given boundary conditions and require that the extremal originate from a given initial point B and terminate in a fixed point E.

First let us consider a family of extremals terminating in a given point E on the lower left side-line of the rectangle. Fig. 3. represents some of these extremals calculated by means of a digital computer.[6])

If the prescribed initial point B is located under the total extremal the corresponding extremal originating in B at first approaches the total

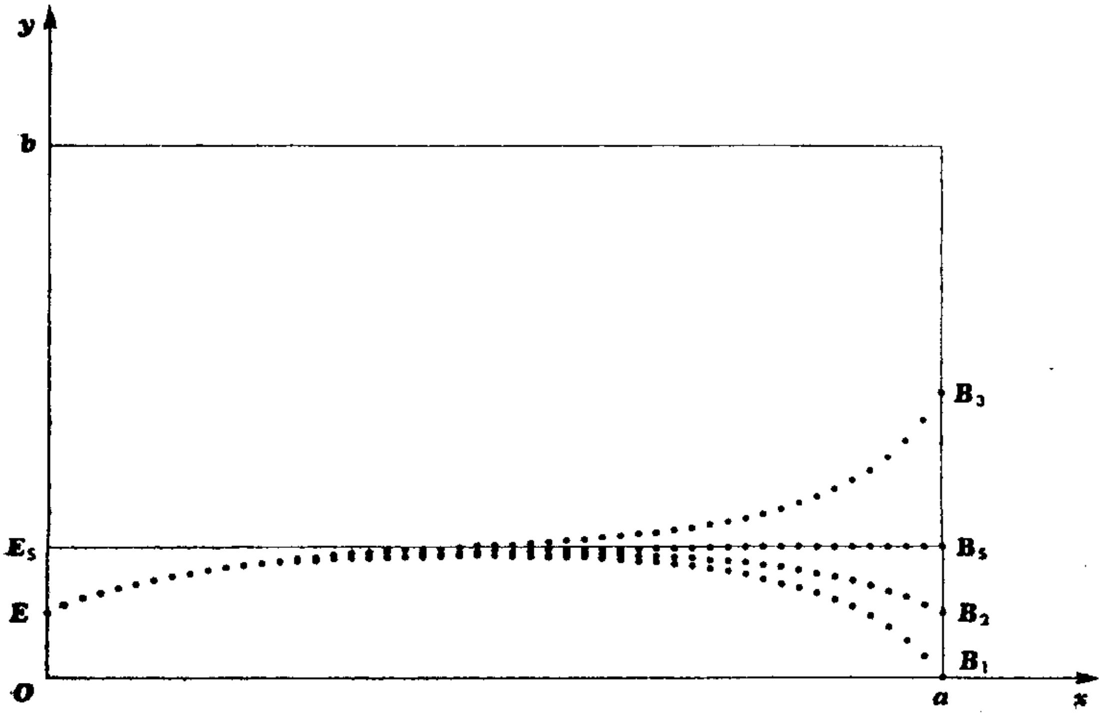

Fig. 3. A Family of Extremals

[6]) Fig. 3 represents the extremals terminating in a common point E (0,150) and originating from points B_1 (2000,0), B_s (2000,150), B_2 (2000,300), B_3 (2000,750), respectively.

extremal, achieves a maximum in a certain point under it and then departs from the total extremal in order to reach the prescribed terminating point E. If the prescribed initial point B is located above the total extremal, the corresponding extremal at first approaches the total extremal, presently intersects it at a certain point and then departs from the total extremal in order to reach the terminating point E. The calculated flow of extremals corresponds with the intuitive discernment that the extremal, while fulfilling the boundary conditions, tends to approach the total extremal as close as possible.

Among all the extremals terminating in a prescribed point E let us determine the one for which the compound transportation costs are minimal. It shall be seen that this is a distinct property of the extremal originating from the point B_s at the beginning of the total extremal.

First let us convince ourselves of the fact that, in comparison with any other extremal originating in some point B above B_s the compound transportation costs corresponding to the extremal from B_s to E are minimal; the basis of this reasoning is represented on fig. 4.

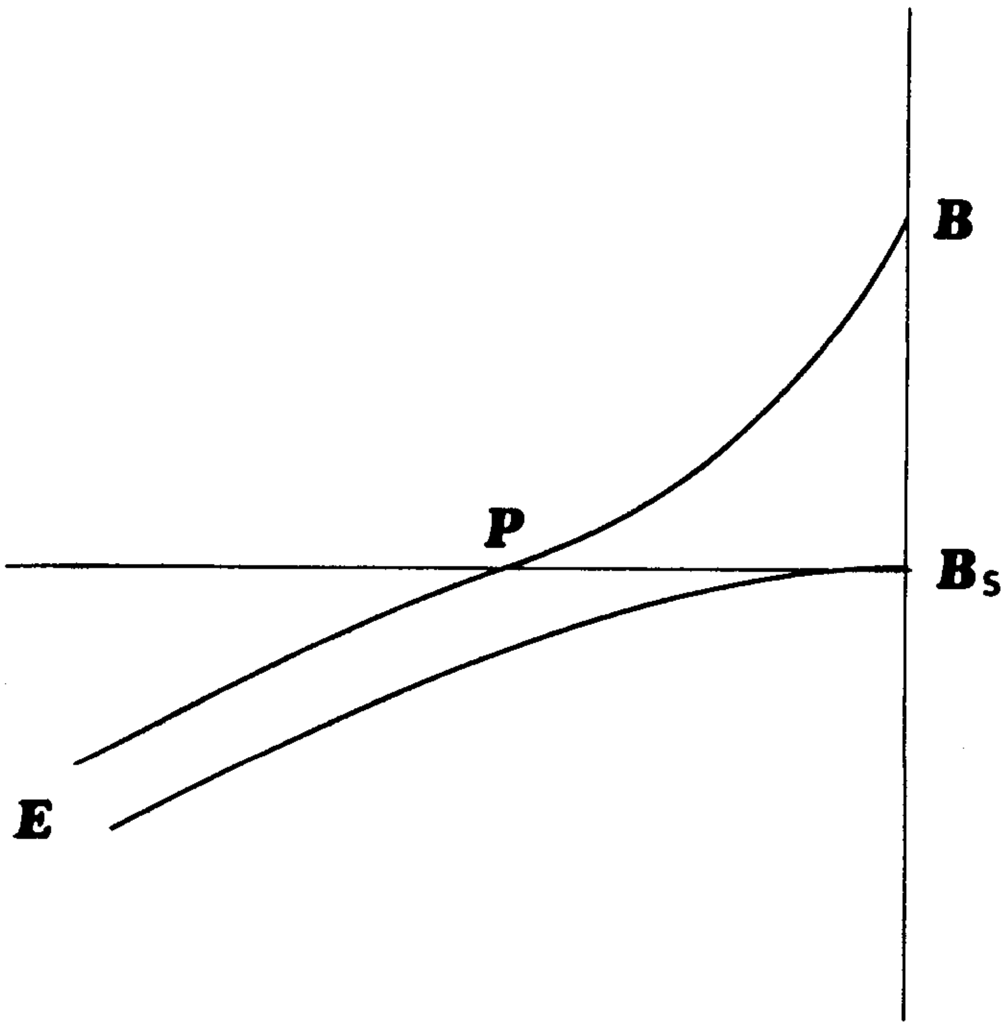

Fig. 4. Comparison of Extremals

Let P be the intersection point of the extremal from B to E with the total extremal. The compound transportaion costs $T(B, P, E)$ along the extremal through B, P and E are greater than the compound transportaton costs $T(B_s, P, E)$ along the route through B_s, P and E:

$$T(B, P, E) > T(B_s, P, E).$$

This results from the fact that the route from B to P is longer, and from the viewpoint of transportation costs as well, less convenient then the route along the total extremal from B_s to P. Furthermore, the compound transportation costs $T(B_s\ P, E)$ along the route through B_s, P and E are greater then the compound transportation costs $T(B_s, E)$ along the extremal from B_s to E:

$$T(B_s, P, E) > T(B_s, E)$$

This follows from the fact that the route from B_s to E is an extremal and therefore the corresponding compound transportation costs are lesser in comparison with any other route connecting the two points. Both-inequalities yield:

$$T(B, E) > T(B_s, E),$$

This inequality applies in cases where the point B is located above the total extremal. Presently, let us establish that the compound transportation costs corresponding to the extremal from B_s to E are less than those corresponding to any other extremal originating in a certain point B located under the total extremal. In order to accomplish this we have to prove that for each extremal passing through the point B located under B_s there exists at least one extremal for which the corresponding compound transportation costs are less; this line of reasoning is illustrated on fig. 5.

Let the extremal from B to E reach its maximum at the point M located between B and B_s on the total extremal: To the point M is assigned a certain point B_1 located on the right side-line of the rectangle and having the same ordinate as M; therefore B_1 is located between B and B_s.

The compound transportation costs $T(B, M, E)$ along the extremal through B, M and E are greater then the compound transportation costs $T(B_1, M, E)$ along the route through B_1, M and E:

$$T(B, M, E) > T(B_1, M, E)$$

This follows from the fact that the route from B to M is longer and from the viewpoint of transportation costs less convenient then the route through B_1 and M. Furthermore, the compound transportation costs $T(B_1, M, E)$ along the route through B_1, M and E are greater then the compound transportation costs $T(B_1, E)$ along the extremal from B_1 to E:

$$T(B_1, M, E) > T(B_1, E)$$

This follows from the fact that the route from B_1 to E is an extremal and therefore corresponding transportation costs are less then those for any other route connecting the two points. Utilizing both inequalities yields:

$$\| T(B_1, E) < T(B, E) \|$$

This inequality applies to the situations where the point B_1 is located under B_s and B_1 lies between the points B and B_s. Thus, from the viewpoint of the compound transportation costs we have obtained a better extremal through B_1 and E to the extremal through B and E. Repeating

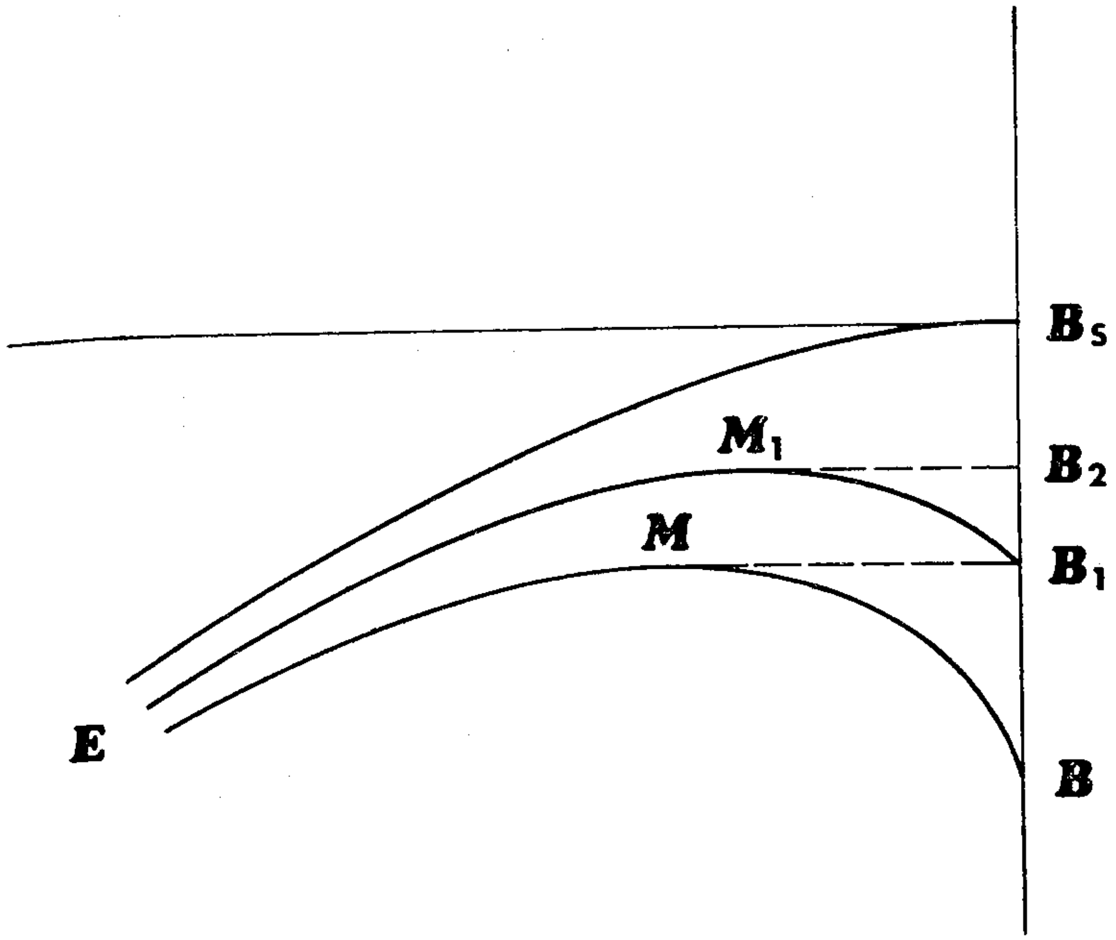

Fig. 5. Comparison of Extremals

the above described procedure we obtain, from the viewpoint of the compound transportation costs, a still better extremal from B_2 to E. Successive repetitions of this procedure renders a sequence af extremals in which each term is better then the former from the viewpoint of the compound transportation costs. All these extremals terminate in the point E and originate from the points:

$$B, B_1, B_2, B_3, \ldots,$$

respectively. Knowledge of the general solution of the differential equation (9) would enable us to prove that the ordinates of these points increase and furthermore, that the sequence comprised of the ordinates of these points converges towards the ordinate of the point B_s. Accordingly it follows that the compound transportation costs corresponding to the extremal from B_s to E are less then those corresponding to any other extremal terminating in the point E and originating in any point B located under the total extremal.

Consequently, among all the extremals terminating in a prescribed point E and originating from a certain point B located under or on the total extremal, it is the extremal from B_s to E for which the corresponding transportation costs are minimal. This extremal originates in the point B_s at the beginning of the total extremal and then descends in order to reach the prescribed terminal point E on the left side-line of the rectangle.

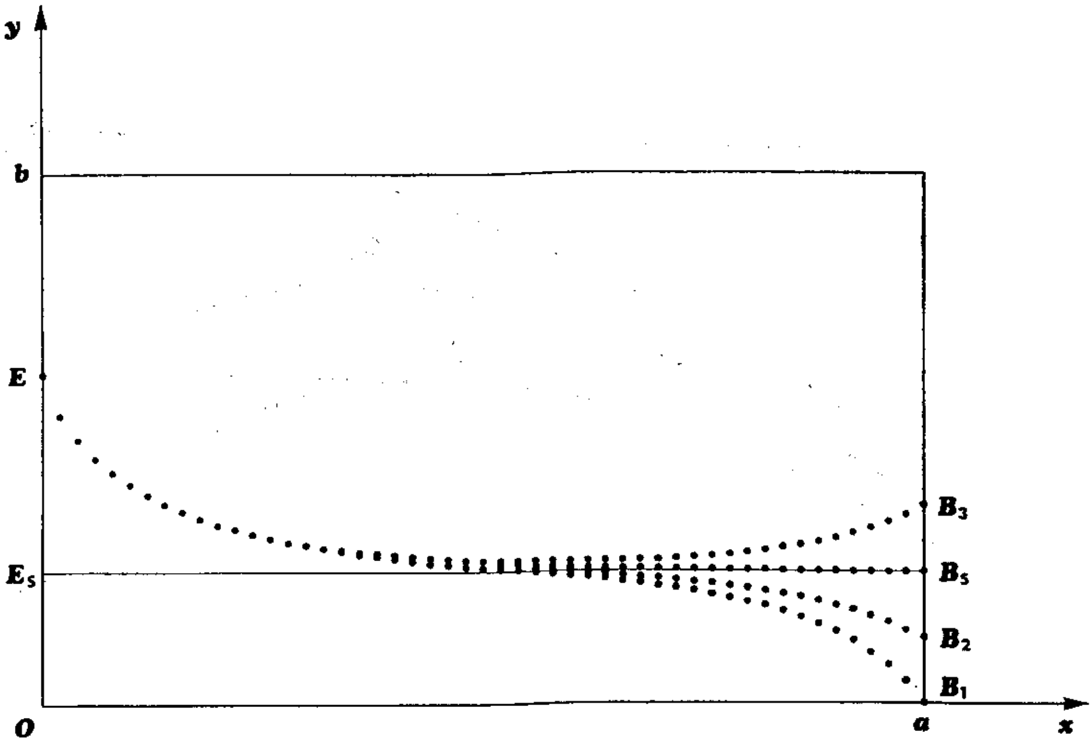

Fig. 6. A Family of Extremals

Similar conclusions apply to cases where the prescribed terminal point E is located above the total extremal. Fig. 6. represents some of these extremals calculated by means of a digital computer.[7]) Among all these extremals it is the extremal from B_s to E for wich the compound transportation costs are minimal. This extremal originates from the point B_s at the beginning of the total extremal and then curves upward in order to reach the prescribed terminal point E. Thus we have obtai-

[7]) Fig. 6. represents a family extremals with a common terminal point E (0,750) and originating from B_1 (2000,0), B_2 (2000,150), B_s (2000,300) and B_3 (2000,450), respectively.

ned the complete solution to the problem of the optimal location of the route terminating in a prescribed point E on the left side-line of the rectangle. Namely, the optimal route in each particular case is determined by the extremal originating from the point B_s at the beginning of the total extremal and termnating in a prescribed point E. Fig. 7. represents some of these extremals calculated by means of a digital computer.[8])

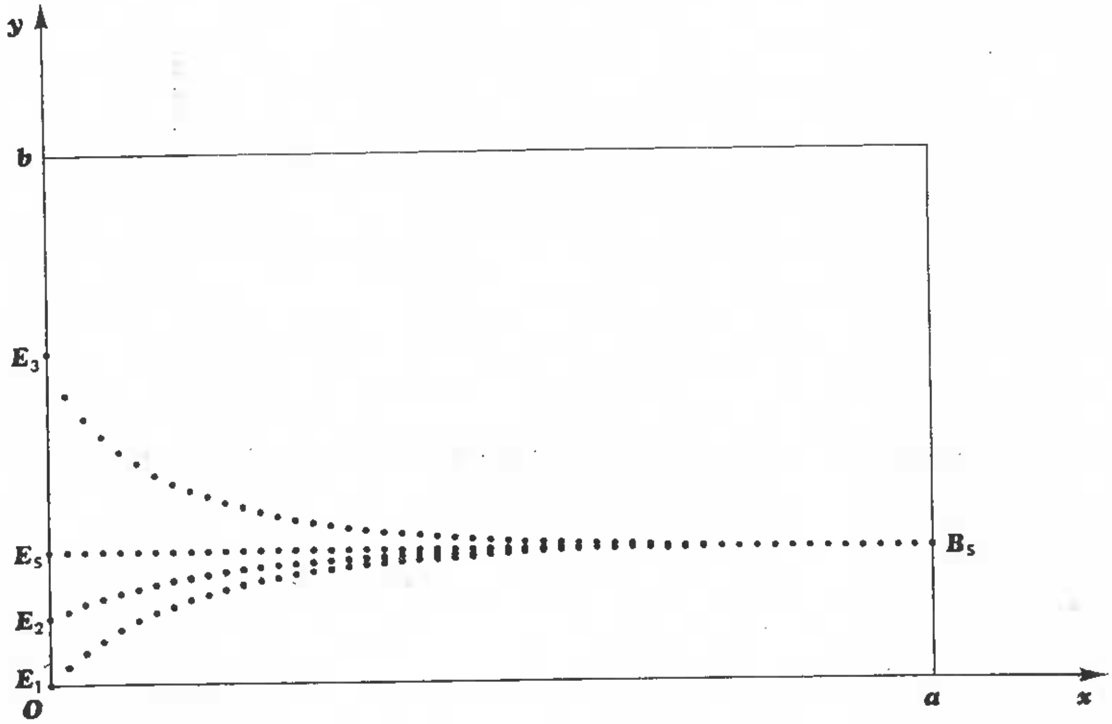

Fig. 7. Optimal Routes

All these extremals originate from the point B_s at the end of the total extremal and then depart from it in order to reach the prescribed terminal point E on the left side-line of the rectangle.

Presently let us take into account some interesting possibilities to be encountered when the parameters assume certain exceptional particular values.

First, let us consider the case where $p_4 = 0$, i. e. where the costs for the artificial route are omitted. In this case the differential equation (9) has a particular solution:

This can be readily verified by substitution into the differential equation. In this particular case the optimal route originates in the initial point

$$y = \frac{p_1}{p_1 + p_2} b = s$$

[8]) Fig. 7. represents a group of extremals with a common initial point B_s (2000,300) and terminating in E_1 (0,0), E_2 (0,150), E (0,300) and E_3 (0,750), respectively.

B_s, follows the total extremal to the point E_s on the left side-line of the rectangle and then follows the side-line in order to reach the prescribed terminal point E. This case is represented on fig. 8.; the point E is located under the total extremal.

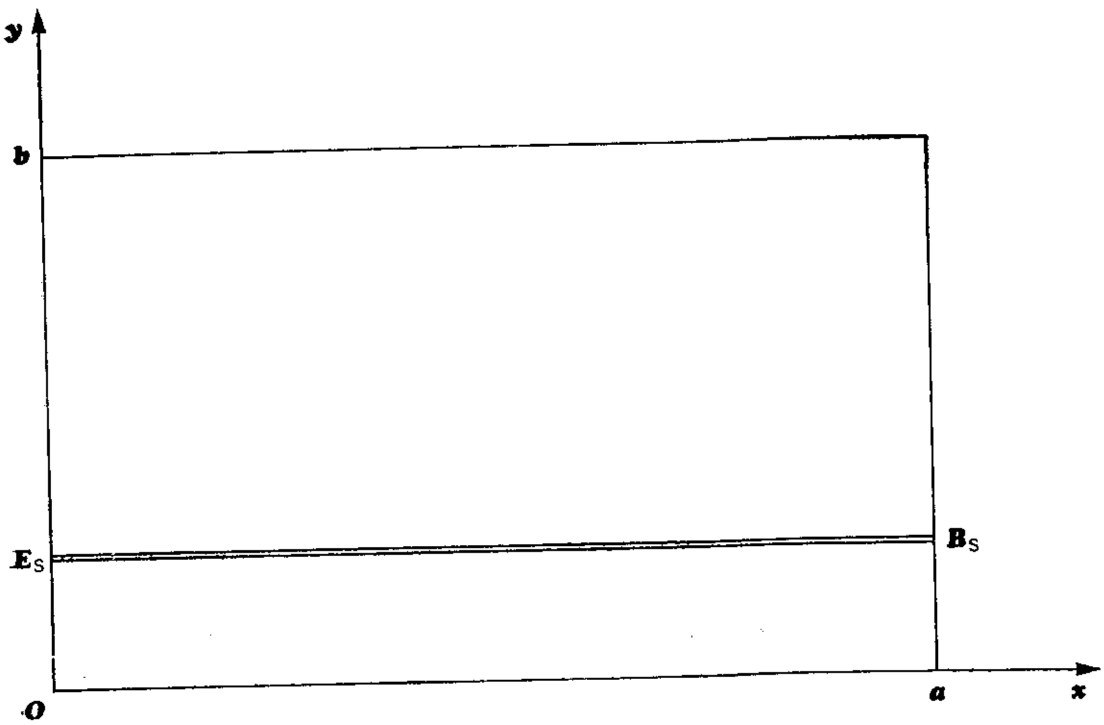

Fig. 8. An Optimal Route where $p_4 = 0$.

Another possibility arises when $p_4 = \infty$, i. e. all other costs are negligibly small compared to the costs for the artificial route. Dividing the differential equation (9) and neglecting the terms having p_4 in the divisor, we obtain the differential equation:

$$\frac{y''}{\sqrt{(1 + (y')^2)^3}} = 0$$

Since the divisor on the left side of the equation does not equal 0 we obtain the differential equation of second order:

$$y'' = 0$$

with the general solution:

$$y = c_1 x + c_2$$

The constants c_1 and c_2 can be determined utilizing the condition that the route connects B_s with the prescribed terminal point E. In this case the optimal route is a straight line through B_s and E. This route is represented on fig. 9.

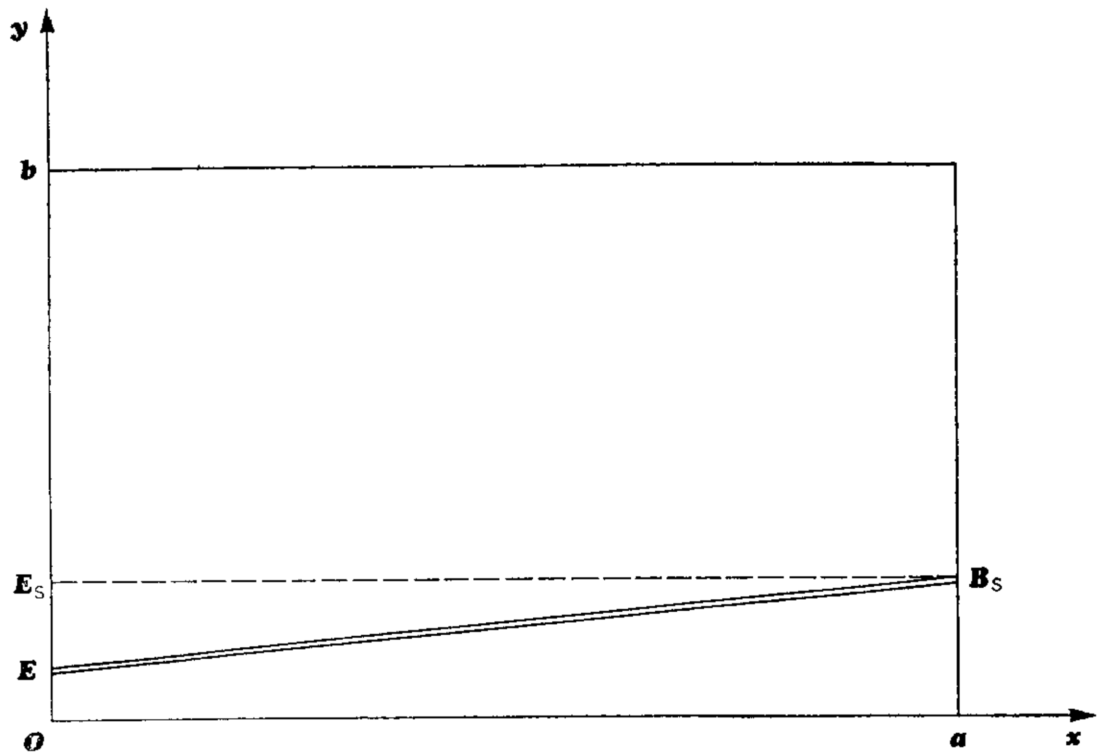

Fig. 9. The Optimal Route where $p_4 = \infty$

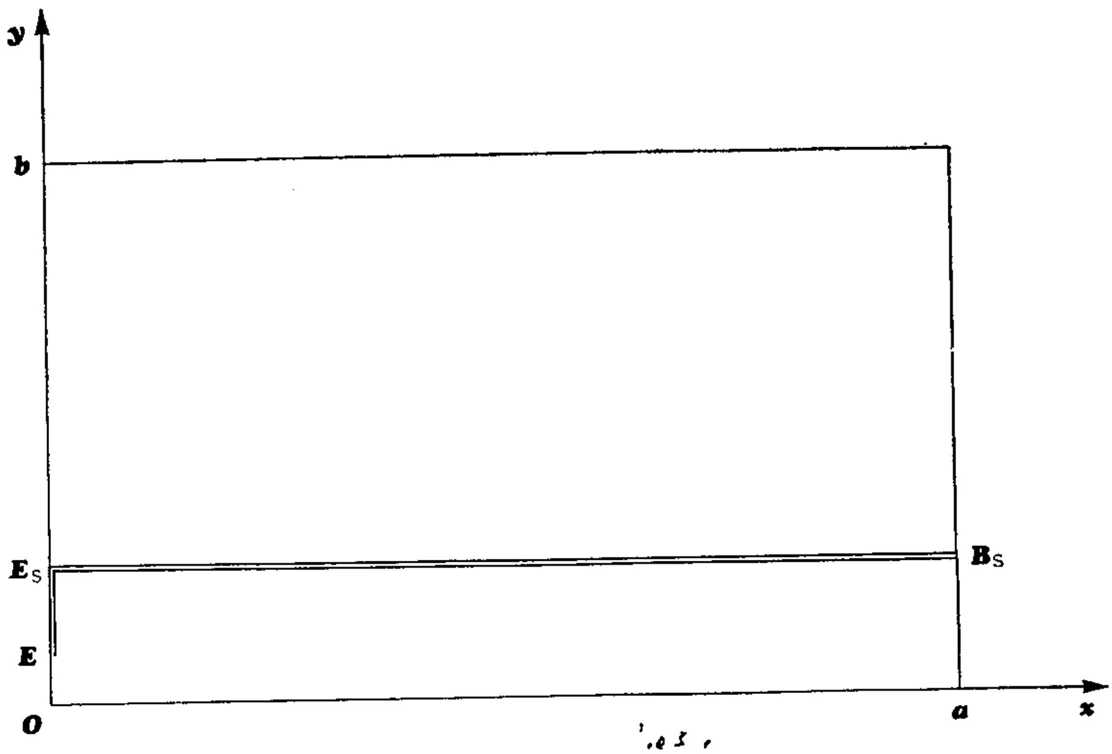

Fig. 10. The Optimal route where $p_2 = \infty$

Finally let us consider the possibility when $p_2 = \infty$, i. e. transport upward to the artificial route cannot be taken into account. Dividing the differential equation (9) by p_2 and omitting the terms having p_2 in the divisor renders the equation:

$$y = 0$$

In this case the optimal route follows the base-line of the rectangle from the left side-line to the origin and then follows the right side-line in order to reach the prescribed terminal point E. Such a route is represented on fig. 10.

LOKACIJA OPTIMALNE POTI NA PRAVOKOTNI PLOSKVI

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Rezime

V članku je rešen problem optimalne lokacije ene poti na pravokotni površini; pri tem vodi pot, kakor se vidi na sl. 1., od desnega roba do predpisane točke E na levem robu pravokotnika. Kriterij optimalnosti je zahteva, da naj bodo skupni prevozni stroški najmanjši.

Določevanje optimalne trase poti je obravnavano kot problem variacijskega računa, ki privede do reševanja dokaj komplicirane diferencialne enačbe drugega reda (3). Nekatere posebne rešitve te diferencialne enačbe, ki zadoščajo določenim robnim pogojem, so izračunane po točkah z elektronskim računalnikom. Optimalno traso določuje v vsakem primeru ekstremala, ki začne, kakor vidimo na sl. 8., v skupni začetni točki B_s in konča v kaki predpisani končni točki E.