PROGRAMMING MULTI-PHASE PROCESSES

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I. INTRODUCTION

By a multi-phase process we mean a process which can be, with regard to time, divided into several successive phases, developing with the time one after another, and where the results of each previous phase may have an effect on each later one.

Agricultural production, e.g., can be considered a two-phase process, as it occurs in two phases: in the first phase fodder is grown on certain cultivable lands under certain agrotechnical conditions, which is used, in the second phase, for breeding domestic animals.

A furhter example of a two-phase production is industrial production in which machines, as intermediate products, are first produced in the first phase, by which in the second phase, finished articles are produced.

Multi-phase production problems are very often met with in production enterprises which are not vertically specialized and in which, for whatever reasons, they try to do all the operations »at home«, starting with raw materials, through phase-articles of various degrees, to finished articles. In programming multi-phase process there arises the question of how to do the entire process in order to achieve the optimum from the point of view of the determined, chosen in advance criterion. In it we confine ourselves to such types of the multi-phase processes that lead us, when finally formulated mathematically, to the problems of linear programming.

Each multi-phase process can be decomposed into several successive one-phase processes. In each of such a one-phase process there occur, as parametres, the results of the previous phases. All these individual phases should be programmed in a way so as to achieve the optimum in the entire multi-phase process.

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II. FLOW CHART OF A TWO-PHASE PRODUCTION PROCESS

Each two-phase production process can be first roughly decomposed into two phases. In the first phase we produce, starting from the monetary means available and from various elements of the production process, the phase products and then, in the second phase, the finished articles. For the needs of the quantitative analysis of the production process, however, such a decomposition is too rough. For these reasons we shall, in our further research, decompose the production process in a more detailed way and illustrate it in such a chart of flow that will include all possible various decisions and possibilities that may occur in a two-phase production process. The entire chart of flow is shown in the Figure.

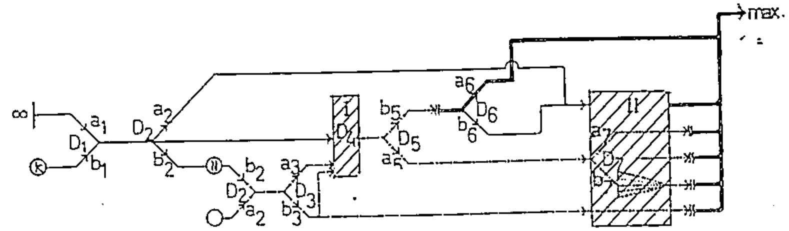

Figure. Flow chart of a two-phase production process.
———— *monetary means*
— — — — *elements of the production process*
— . — . — *phase articles*
. . . . . . *finished articles*

Decision D_1. Monetary means are needed to carry out the entire production process. With regard to the quantity of the monetary means available, it is necessary to evaluate and decide which of the two alternatives exists: whether the monetary means are unlimited (alternative a_1), or the monetary means are limited (alternative b_1).

Alternative a_1. In this alternative the monetary means available are unlimited, which practically means that they are available in such a quantity that is necessary for any technical carrying out of a technological process. As the monetary means in this alternative are practically unlimited we assume that they amount to:

$$K_1 = \infty$$

In this alternative we have to deal with a completely technical production in which costs are not at all taken into account; such a production is determined only by material and technical capacities.

As in this alternative the expenditure of monetary means need not be taken into account in the further production process either, only the income derived from the articles sold is taken into account in the final objective function. Thus, in the chart of flow of the production process only that part of the flow of the monetary means is taken into account for determining the objective function, which is marked with a double continual line. In this alternative we wish to obtain the maximum income from the articles sold.

Alternative b_1. In this alternative the monetary means which can be invested in the production process, are limited and amount, in corresponding monetary units, to

$$K_1 = k.$$

In this alternative production depends on the quantity of the invested monetary means, as well as on the material and technical capacities. As, in this alternative, in the entire further process of production, the inflow and outflow of the monetary means have to be taken into account, it is also necessary to take into account, in the final objective function, the final balance of the monetary means. Thus, in the flow chart of this alternative the entire drawn flow of the monetary means comes into account. In this alternative we wish to achieve the maximum final balance of the monetary means.

Decision D_2. Various elements of the production process are put into the production process, such as various raw materials, differently skilled labour, etc. The elements of the production process participating in the production process are successively marked with:

$$Q_1, \ldots, Q_i, \ldots, Q_s.$$

There are only limited quantities of the individual elements of the production process available; these quantities available are shown in the corresponding units in the matrix:

$$S = \left\| \begin{matrix} s_1 \\ \cdot \\ \cdot \\ s_i \\ \cdot \\ \cdot \\ s_s \end{matrix} \right\|$$

Purchase prices correspond to the individual elements of the production process; they are shown in the matrix:

$$U = \| u_1, \ldots u_i, \ldots u_s \|.$$

Certain quantities of the elements of the production process are put in the production process. They are shown in the matrix:

$$S_2 = \left\| \begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_i \\ \cdot \\ \cdot \\ z_s \end{matrix} \right\| = Z.$$

As the available quantities of the elements of the production process are limited, the quantities put in correspond to the matrix inequality:

$$S_2 \leqq S. \qquad (2,S)$$

With regard to the purchase of the elements of the production process it is necessary to evaluate and decide whether the elements of the production process are directly available and it is not necessary first to purchase them (alternative a_2), or they have to be purchased at the purchase prices (alternative b_2).

Decision D_2 requires automatically also a corresponding decision about investing the monetary means in the purchase of the elements of the production process. If it is not necessary to purchase the elements of the production process, then it is not necessary to spend the monetary means (alternative a_2); if, however, it is necessary to purchase the elements of the production process, then it is necessary to spend some monetary means to the purpose. (alternative b_2).

Alternative a_2. In this alternative the elements of the production process are directly available and it is not necessary first to purchase them. For this reason, in this alternative, we formally assume that the purchase price of all the elements of the production process equals 0, thus:

$$U = 0.$$

As in this alternative monetary means are not spent, they amount, after the assumption of the alternative a_2, to:

$$K_2 = K_1.$$

The quantities of the elements of the production process, put into the production, amount to:

$$S_2 = Z.$$

Alternative b_2. In this alternative the elements of the production process have to be purchased at the determined purchase prices which are positive:

$$U > 0.$$

In the purchase of the $S_2 = Z$ elements of the production process we spend

$$u_1 z_1 + \ldots + u_i z_i + \ldots + u_s z_s = U S_2$$

monetary means so that there remain, after the realization of this alternative

$$K_2 = K_1 - U.S_2 = k - U\ S_2$$

monetary means available. As the remainder of the monetary means cannot be a negativne one, it is possible only to purchase such quantities of the elements of the production process that the inequality is true:

$$k - U\ S_2 \geqq 0. \qquad (2,K)$$

Decision D_3. With regard to the terms of putting in the elements of the production process we have to evaluate and to decide, whether all the available quantities of the elements of the production process should be put into the first phase directly (alternative a_3), or they should be put partly into the first phase and partly preserved for the second one (alternative b_3).

Alternative a_3. In this alternative all the quantities of the elements of the production process are entirely put into the first phase and are not needed in the second one. There is a matrix in this alternative equaling ϕ, which is marked with letter B and which will be defined later on. As we have to deal in this alternative, only with the consecutive order of the elements of the production process, there is no change in the quantity of the monetary means available, nor in the quantity of the available elements of the production process. Owing to it the quantity of monetary means, K_3 remaining available after the assumption of this alternative, equals the quantity of the previously available monetary means K_2:

$$K_3 = K_2.$$

For the same reason, the following equation is true also for the quantities of the available elements of the production process:

$$S_3 = S_2.$$

Alternative b_3. In this alternative the elements of the production process are partly put into the first phase and partly preserved for the production in the second phase. In this alternative the above-mentioned matrix B is a positive one. As this alternative deals only with the consecutive order of the elements of the production process, without changing any quantities, the following equations are true:

$$K_3 = K_2.$$
$$S_3 = S_2.$$

First Phase of Production (Decision D_4). In the first phase of production we produce using the monetary means available and the available elements of the production process, the phase articles of vaious types, which are successively marked with:

$$F_1, \ldots\ldots F_j \ldots\ F_f.$$

Let us assume to have, in producing one unit of the phase article, type F_j k_j money units of direct expenses, and that we spend a_{ij} units of the element of the production process Q_i. The quantities of the produced phase articles of the individual types are shown in the matrix

$$X + X^p = \left\| \begin{matrix} x_1 + x_1^p \\ \cdots\cdots \\ x_j + x_j^p \\ \cdots\cdots \\ x_f + x_f^p \end{matrix} \right\|$$

The phase articles have the determined selling prices shown in the matrix

$$V = \| v_1, \ldots, v_j, \ldots, v_f \|.$$

Thus the production process in the first phase is shown in Table 1.

Table 1. First Phase.

	$F_1 \ldots F_j \ldots F_f$	Constraints
K	$k_1 \ldots k_j \ldots k_f$	K_3
Q_1	$a_{11} \ldots a_{1j} \ldots a_{1f}$	z_1
.	. . .	.
Q_i	$a_{i1} \ldots a_{ij} \ldots a_{if}$	z_i
.	. . .	.
Q_s	$a_{s1} \ldots a_{sj} \ldots a_{sf}$	z_s
Quantities	$x_1 + x_1^p \ldots x_j + x_j^p \ldots x_f + x_f^p$	
Prices	$v_1 \ldots v_j \ldots v_f$	

Starting from the Table determining the production process in the first phase, we further define the following matrices:

$$K = \| k_1 \; . \; . \; . \; k_j \; . \; . \; . \; k_f \|,$$

$$A = \begin{Vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1f} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{if} \\ \vdots & & \vdots & & \vdots \\ a_{s1} & \dots & a_{sj} & \dots & a_{sf} \end{Vmatrix} \quad X = \begin{Vmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_f \end{Vmatrix} \quad X^p = \begin{Vmatrix} x_1^p \\ \vdots \\ x_j^p \\ \vdots \\ x_f^p \end{Vmatrix}$$

$$V = \| v_1 \; . \; . \; . \; v_j \; . \; . \; . \; v_f \|.$$

In the first phase of production the expenditure of monetary means amounts to:

$$k_1 \left(x_1 + x_1^p\right) + \; . \; . \; . \; + k_f \left(x_f + x_f^p\right) = K \; (X + X^p);$$

for this reason there remains, after the conclusion of the first phase,

$$K_4 = K_3 - K \; (X + X^p)$$

of monetary means. As non-negativeness is required, there follows the inequality:

$$K_3 - K \; (X + X^p) \geqq 0. \qquad (4,K)$$

As after assuming alternative a_2

$$K_3 = K_2 = k - U \; S_2,$$

the following inequality is derived from the previous equality:

$$k - U \; S_2 - K \; (X + X^p) \geqq 0;$$

as the inequality (2,K) is automatically true if only the last inequality has been true, we may leave it out as a condition in the linear programme. Leaving out the inequality (2,K) is, of course, allowed only in the production process where the alternative a_2 occurs.

In the first phase of production the elements of the production process are used

$$A \; (X + X^p)$$

therefore, there remain, after the conclusion of the first phase:

$$S_4 = S_3 - A \; (X + X^p).$$

As in each case $S_3 = S_2$, we obtain from it, owing to the required non-negativeness for the elements of the production process, the inequality:

$$S_2 - A \; (X + X^p) \geqq 0. \qquad (4,S)$$

Finally, let us establish that there are, after the conclusion of the first phase of production, phase articles produced available

$$F_4 = X + X^p.$$

Decision D_5. With regard to the disposal with the phase articles produced it is necessary to evaluate and decide whether the phase articles will be entirely used in the second phase of production (alternative a_5), or they will be used in part in the second phase and in part sold in the market, thus eliminating them from the production process (alternative b_5).

The quantities of the phase articles, used in the second phase of production have been marked with matrix X, and the quantities to be sold, have been marked with matrix X^p.

Alternative a_5. In this alternative we use all phase articles production, and so we do not sell them at all. For this reason, in this alternative

$$X^p = 0;$$

and, owing to it, the production of phase articles in the first phase amounts to X. As in assuming this alternative, neither the quantity of the monetary means available, nor the quantity of the available elements of the production process are changed, the equations are true:

$$K_5 = K_4,$$
$$S_5 = S_4,$$

and as far as the quantities of phase articles are concerned it is true, as in the previous case:

$$F_5 = F_4 = X.$$

Alternative b_5. In this alternative we use a part of the phase articles in the second phase of production and the remaining part is sold; in this we use X in the second phase and we sell X^p of the phase products.

Owing to the sale of the phase articles the monetary means increase by

$$v_1 \, x_1^p + \; . \; . \; . \; + v_j^p \, x_j + \; . \; . \; . \; + v_f^p \, x_f = V \, X^p$$

monetary units, to

$$K_5 = K_4 + V \, X^p$$

As the quantities of the elements available in the production process remain unchanged in this alternative it is true:

$$S_5 = S_4.$$

Owing to the sale X^p of the phase articles there remain, for further disposal

$$F_5 = X$$

phase articles.

Decision D_6. With regard to the possibility to use the monetary means obtained from the sale of the phase articles, we have to evaluate and decide whether these means should not be used in the second production phase (alternative a_6), or they should be used in the second phase (alternative b_6).

Alternative a_6. In this alternative we do not use the monetary means obtained from the sale of the phase articles in the second phase. The monetary means obtained in this way are eliminated from the further production process and they are taken into account only at the conclusion when drawing up the balance sheet. As these monetary means are not taken into account further on, the previous monetary means available K_5 have to be reduced by the proceeds obtained.

Owing to it the monetary means available for further production process amount to:

$$K_6 = K_5 - V X^p = K_4.$$

As, in this alternative, neither the quantities of the available elements of production process, nor the quantities of the available phase articles are changed, the following equations are true:

$$S_6 = S_5,$$

$$F_6 = F_5 = X.$$

Alternative b_6. This alternative includes the monetary means obtained from the sold phase article and the existing monetary means that are to be used together in production in the second phase. As, owing to this, the monetary means available are not changed, they amount, after the realization of this alternative to:

$$K_6 = K_5 = K_4 + V X^p .$$

As in this alternative neither the available quantities of the elements of the production process, nor those of the phase articles are changed, the equations are true:

$$S_6 = S_5,$$

$$F_6 = F_5 = X.$$

Second production phase (Decision D_7*).* In the second phase we produce starting from the available quantitites of the monetary means, of the elements of the production process and of the phase articles, the warious types of finished articles which are successively marked with

$$P_1, \ldots, P_k, \ldots, P_p .$$

With regard to the way of spending the phase articles as factors of production, two alternatives are distinguished in the second phase. In the first alternative (a_7) the phase articles are disappearing during the production of finished articles, being, in a given technological process, somehow changed or transformed into the finished articles; we have to deal with such an alternative, e.g. in the two-phase agricultural production where, in the first phase, we grow fodder which is then, in the second phase, used for breeding animals, these being the finished products. In the second alternative (b_7), the phase articles are wearing out during the production in the second phase thus diminishing their value; we have to deal with such an alternative, e.g. in industrial production, where, in the first phase, we produce machines as intermediate articles and then, in the second phase, by the means of them, the finished articles.

All coefficients, determining the production in the second phase, are shown in Table 2.

Table 2. Second Phase.

	P_1	. . .	P_k	. . .	P_p	Constraints		
K	h_1	. .	h_k	. . .	h_p	K_6		
Q_1	b_{11}	. .	b_{1k}	. .	b_{1p}			
.	.		.		.			
Q_l	b_{l1}	. .	b_{lk}	. . .	b_{lp}	S_6		
.	.		.		.			
Q_s	b_{s1}	. . .	b_{sk}	. . .	b_{sp}			
F_1	c_{11}	. . .	c_{1k}	. . .	c_{1p}	x_1	r_1	d_1
.	.		.		.	.	.	.
F_j	c_{j1}	. . .	c_{jk}	. . .	c_{jp}	x_j	r_j	d_j
.	.		.		.	.	.	.
F_f	c_{f1}	. . .	c_{fk}	. . .	c_{fp}	x_f	r_f	d_f
Quantities	y_1	. . .	y_k	. . .	y_p			
Prices	w_1	. . .	w_k	. . .	w_p			

Starting from Table 2 which determines the second phase of the production process we can define the following matrices:

$$H = \| \, h_1 \; \ldots \; h_k \; \ldots \; h_p \, \| ,$$

$$B = \begin{Vmatrix} b_{11} & \dots & b_{1k} & \dots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{l1} & \dots & b_{lk} & \dots & b_{lp} \\ \vdots & & \vdots & & \vdots \\ b_{s1} & \dots & b_{sk} & \dots & b_{sp} \end{Vmatrix}$$

$$C = \begin{Vmatrix} c_{11} & \dots & c_{1k} & \dots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{j1} & \dots & c_{jk} & \dots & c_{jp} \\ \vdots & & \vdots & & \vdots \\ c_{f1} & \dots & c_{fk} & \dots & c_{fp} \end{Vmatrix}$$

$$Y = \begin{Vmatrix} y_1 \\ \vdots \\ y_k \\ \vdots \\ y_p \end{Vmatrix}$$

$$W = \| w_1 \ \dots \ w_k \ \dots \ w_p \| ,$$

$$R = \| r_1 \ \dots \ r_j \ \dots \ r_f \| ,$$

$$D = \| d_1 \ \dots \ d_j \ \dots \ d_f \| .$$

Alternative a_7. In this alternative the phase articles disappear during the production process and are somehow transformed into the finished articles. The coefficients occurring in Table 2, determining the second production phase, have in this alternative, the following meaning:

h_k defines the quantity of direct monetary costs spent in the production of one unit of the finished article, of the type P_k

b_{lk} defines the number of units of the element of the production process Q_l spent in the production of one unit of the finished article P_k.

c_{jk} defines the number of units of phase article F_j spent in the production of one unit of the finished article P_k.

y_k defines the number of units of the finished article P_k made in the second phase.

w_k defines the price of the finished article P_k.

Coeficients r_j and d_j do not come into consideration in this alternative.

In the production of Y finished articles

$$h_1 y_1 + \dots + h_k y_k + \dots + h_p x_p = H X$$

units of monetary means are spent. On account of it, after the conclusion of the second phase, there remain

$$K_7 = K_6 - H Y$$

monetary means. As non-negativeness is required the following inequality is true:

$$K_6 - H Y \geqq 0. \qquad (7,K)a)$$

In the production of Y finished articles we spend

$$BY$$

elements of production process. Owing to this there remain after the conclusion of the second phase

$$S_7 = S_6 - B Y$$

elements of production process. As non-negativeness is required the following inequality is true:

$$S_6 - B Y \geqq 0. \qquad (7,S)$$

In the production of Y finished articles we spend

$$CY$$

phase articles. Owing to this, there remains, after the conclusion of the second phase,

$$F_7 = F_6 - C Y$$

phase articles. As non-negativeness is required the following inequality is true:

$$F_6 - C Y \geqq 0. \qquad (7,F)$$

After the conclusion of the second production phase there remain the following quantities available:

1. The monetary means. With regard to the monetary means two elements have to be taken into account: first, the monetary means remaining after the conslusion of the second phase (K_7) and secondly, the possible monetary means obtained from the sale of the phase articles and not used in the second phase, i.e. the means obtained in the assumption of the alternatives b_5 and a_0; these means amount to VX^p. So the total monetary means amount to

$$K_8 = K_7 + V\, X^p\,.$$

2. The elements of the production process. After the conclusion of the second phase there remain S_7 of them, their monetary value amounting to

$$U\, S_7$$

monetary units.

3. The phase articles. After the conclusion of the second phase there remain F_7 of them, their value amounting to

$$V\, F_7$$

monetary units.

4. The finished articles. In the second phase we produce Y of them, their value amounting to

$$W\, Y$$

monetary units.

The total value of all the four types of the quantities mentioned amounts to

$$f = K_8 + U\, S_7 + V\, F_7 + W\, Y; \qquad (8,a)$$

and this value is the objective function, the maximum of which has to be determined.

In this alternative there arises, therefore, the following problem of linear programming for the considered two-phase production process.

To determine the variables Z, X, X^P and Y, satisfying the conditions of the non-negativeness:

$$Z \geqq 0,$$
$$X \geqq 0,$$
$$X^p \geqq 0,$$
$$Y \geqq 0$$

and the inequalities:

$$Z \leqq S, \qquad (2,S)$$
$$(k - U\, Z \leqq 0), \qquad ((2,K))$$
$$K_3 - K\,(X + X^p) \geqq 0, \qquad (4,K)$$
$$Z - A\,(X + X^p) \geqq 0, \qquad (4,S)$$
$$K_6 - H\, Y \geqq 0, \qquad (7,K,a)$$
$$S_6 - B\, Y \geqq 0, \qquad (7,S)$$
$$F_6 - C\, Y \geqq 0 \qquad (7,F)$$

so that the objective function

$$f = K_8 + U\, S_7 + V\, F_7 + W\, Y \qquad (8,a)$$

has the maximum.

Alternative b_7. In this alternative the phase articles are worn out in the second phase and owing to this their value is diminished. This is, for instance, the case when phase articles are machines by the means of which we have to produce finished anticles in the second phase. Using this example and in order to express ourselves in a shorter way we shall, in our further research, call these phase articles just machines. Coefficients shown in Table 2, determining the second production phase, have in this alternative the following meaning:

h_k defines the quantity of direct costs spent in the production of one unit of the finished article of type P_k; this quantity, however, does not include the costs incurred in the work of machines and which are separately taken into account later on.

b_{ik} defines the number of units of the element of production process Q_i, spent in production of one unit of the finished article P_k.

c_{jk} shows the number of time units necessary in the work of the machine F_j for one unit of the finished article P_k.

d_j defines direct costs incurred with the machine, type F_j, when it works one time unit.

r_j is called the coefficient of depreciation of the machine F_j and shows by how many monetary units it works one time unit. In our further research we assume that the total depreciation of the machine is in a direct relation to the time of its work.

y_k shows the number of units of the finished article P_k made in the second phase.

w_k defines the price of the finished articles P_k. The monetary costs, incurred during the second production phase, consist in this alternative, of two parts. The first one includes the costs incurred owing to the coefficients h_k; in the production Y of the finished articles these costs amount to

$$h_1 y_1 + \ldots + h_k y_k + \ldots + h_p y_p = H\, Y$$

monetary units. The second part includes the costs incurred owing to the work of machines and coming from the coefficients c_{jk} and d_j; in the production of one unit of the finished article, type P_k, they amount to

$$d_1 c_{1k} + \ldots + d_j c_{jk} + \ldots + d_f c_{fk}$$

monetary units; owing to this these costs amount, in the production of Y of the finished articles, altogether to

$$DCY$$

monetary units. By adding up both costs we have the total costs for the second phase

$$H\,Y + D\,C\,Y = (H + D\,C)Y$$

monetary units. The monetary means available remaining after the conclusion of the second production phase thus amount to

$$K_7 = K_6 - (H + D\,C)\,Y$$

monetary units. As non-negativeness is required the following inequality for the monetary means is true:

$$K_6 - (H + D\,C)\,Y \geqq 0. \qquad (7,K,b)$$

In the production of Y in the second phase $B\,Y$ elements of the production process are spent. Owing to this, there remain, after the conclusion of the second phase,

$$S_7 = S_6 - B\,Y;$$

as this quantity cannot be negative, the following inequality is true for the production process:

$$S_6 - B\,Y \geqq 0. \qquad (7,S)$$

Let us carry out the constraints due to the limited extent of the phase articles available.

The phase articles of the types:

$$F_1, \; \ldots \; F_j, \; \ldots \; F_f$$

as machines in the second phase are worn out and owing to it their value is depreciated. A phase article is completely worn out when its depreciated value equals ϕ.

As the phase articles of the individual types have successively their initial values;

$$\nu_1, \; \ldots, \; \nu_j, \; \ldots, \; \nu_f$$

monetary units and as their depreciation coefficients are successively equal to:

$$r_1, \; \ldots \; r_j, \; \ldots \; r_f$$

monetary units, the duration of the phase articles is successively

$$t_1 = \frac{\nu_1}{r_1}, \ldots, t_j = \frac{\nu_j}{r_j}, \ldots, t_f = \frac{\nu_f}{r_f}$$

time units. With these numbers showing the duration of the individual phase articles, such as machines, we make the diagonal matrix:

$$T = \left\| \begin{matrix} t_1 & \ldots & 0 & \ldots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \ldots & t_j & \ldots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \ldots & 0 & \ldots & t_f \end{matrix} \right\|$$

There are phase articles such as machines, of the individual types in the second phase successively available in

$$x_1, \; \ldots \; x_j, \; \ldots \; x_f$$

units. Thus there are, for each type of the machines for making the finished articles successively available

$$t_1\,h_1, \; \ldots \; t_j\,x_j, \; \ldots, \; t_f\,x_f$$

time units. After introducing the matrix T these time units are the component parts of the products $T\,X$.

For making Y finished articles $C\,Y$ time units of work are necessary; as the time, necessary for making them cannot exceed the time available, the following inequality is true for the phase articles:

$$C\,Y \leqq T\,X. \qquad (7,F)$$

After the conclusion of the second production phase there are the following quantities available:

The monetary means. In this alternative, too, two component parts have to be taken into account: first, the monetary means remaining available after the conclusion of the second phase (K_7) and the possible monetary means obtained from the sale of the phase articles (VX^p); thus the total monetary means amount to:

$$K_6 = K_7 + V\,X^p.$$

The elements of the production process. There remain, after the conclusion of the second phase S_7, their value amounting to $U\,S_7$ monetary units.

The finished articles. They are Y produced, their value amounting to WY monetary units.

$$V\,X - R\,C\,Y$$

The phase articles. The phase articles, such as machines, are depreciated in the second phase, their depreciated value amounting, at the conclusion of the second phase, to:

monetary units, is calculated in the following way. First, we calculate the working time for the machine of each type. In the second phase we produce y_k finished articles of the type P_k; these articles are made, among others, also with X_j machines of the type F_j. Each machine of the type F_j thus works

$$\frac{y_k}{x_j}$$

units of the article of the type P_k. The corresponding numbers of all the finished articles and of all the machines are shown in Table 3.

Table 3. *The quantitative production of machines*

The number of products of the type						
		P_1	...	P_k	...	P_p
worked by one machine of the type	F_1	$\frac{y_1}{x_1}$	...	$\frac{y_k}{x_1}$	...	$\frac{y_p}{x_1}$
	.	.		.		.
	F_j	$\frac{y_1}{x_j}$	...	$\frac{y_k}{x_j}$	...	$\frac{y_p}{x_j}$
	.	.		.		.
	F_f	$\frac{y_1}{x_f}$	...	$\frac{y_k}{x_f}$	...	$\frac{y_p}{x_f}$

Table 3 shows that each machine of the type F_j works

$$\frac{y_1}{x_j}, \ \dots \ \frac{y_k}{x_j}, \ \dots \ \frac{y_p}{x_j}$$

articles of the type

$$P_1, \ \dots \ P_k, \ \dots \ P_p.$$

To do this work the machine F_j has to work

$$c_{j1}\frac{y_1}{x_j} + \ \dots \ + c_{jk}\frac{y_k}{x_j} + \ \dots \ + c_{jp}\frac{y_p}{x_j}$$

time units. As the depreciation coefficient of this machine is r_j, it depreciates for

$$\frac{r_j}{x_j}(c_{j1}y_1 + \ \dots \ + c_{jk}y_k + \ \dots \ + c_{jp}y_p)$$

monetary units, so that its final, depreciated value amounts to

$$v_j - \frac{r_j}{x_j}(c_{j1}y_1 + \ \dots \ c_{jk}y_k + \ \dots \ + c_{jp}y_p)$$

monetary units. Owing to this the depreciated value of all x_j machines, of the type F_j amounts to

$$v_j x_j - r_j(c_{j1}y_1 + \ \dots \ + c_{jk}y_k + \ \dots \ + c_{jp}y_p)$$

monetary units. By adding up the depreciated values of all the machines, of all the types, we get their value after the conclusion of the second phase, amounting to:

$$\sum_{j=1}^{j=f} v_j x_j - \sum_{j=1}^{j=f} r_j(c_{j1}y_1 + \ \dots \ + c_{jk}y_k + \ \dots \ + c_{jp}y_p) = VX - RCY.$$

which was to be calculated.

It is certain that the calculated value

$$VX - RCY$$

of the depreciated phase articles is not a negative one. This can be found out in the following way: The inequality (7,F,b):

$$TX - CY \geqq 0$$

is premultiplied by matrix R and we have:

$$RTX - RCY \geqq 0;$$

as $RT = V$, we get

$$VX - RCY \geqq 0,$$

which was to be proved.

The final total value of the four mentioned types of the quantities amounts to:

$$f = K_8 + US_7 + (VX - RCY) + WY; \qquad (8,b)$$

and this value is the objective function.

Thus, also in this alternative arises the problem of linear programming, which is formulated in this way:

The variables Z, X, X^p and Y have to be determined satisfying the conditions of non-negativeness

$$Z \geqq 0,$$
$$X \geqq 0,$$
$$X^p \geqq 0,$$
$$Y \geqq 0$$

and of the inequalities:

$$Z \leqq S, \qquad (2,S)$$

$$(k - U\ Z \geqq 0), \qquad ((2,K))$$

$$K_3 - K\,(X + X^p) \geqq 0 \qquad (4,K)$$

$$Z - A\,(X + X^p) \geqq 0, \qquad (4,S)$$

$$K_6 - (H + D\,C)\ Y \geqq 0, \qquad (7,K,b)$$

$$S_6 - B\ Y \geqq 0 \qquad (7,S)$$

$$CY \leqq TX$$

so that the objective function

$$f = K_8 + U\ S_7 + (V\ X - R\ C\ Y) + W\ Y \qquad (8,b)$$

has the maximum.

III. THE CHARACTERISTIC OF THE TWO-PHASE PROCESS

Each two-phase production process of the type dealt with, of which the alternative coming into consideration, is known for each decision, can be marked with a characteristic in which all the corresponding alternatives are successively mentioned. With the characteristic:

$$(b_1,\ b_2,\ a_3,\quad b_5,\ b_6,\ a_7)$$

we characterize, e.g. the production process having, in the corresponding decisions, the following alternatives: b_1 the monetary means are limited, b_2 the elements of the production process have to be purchased at purchase prices; a_3 the elements of the production process are entirely used in the first phase, as they are not necessary in the second one; b_5 the phase articles are partly sold and partly used in the second phase; b_6 the monetary means obtained from the sold articles are included in the monetary means available and used together in the second phase; a_7 the phase articles in the second phase disappear.

If we have to deal with a two-phase production problem of the type dealt with it is sufficient to determine its corresponding characteristic. As soon as it is determined, we formulate the respective linear programme using the calculations corresponding to the individual alternative which have already been made. In making up the linear programme we can make use of Table the 4, where all the necesary results are systematically collected.

In every two-phase production process of the type dealt with at $2^6 = 64$ possibilities with various alternatives could occur, according to

Table 4

	a	b
1	$K_1 = \infty$	$K_1 = k$
2	$K_2 = K_1$ Q $S_2 = Z$ $S_2 \leqq Q$	$K_2 = K_1 - US_2$ Q $S_2 = Z$ $S_2 \leqq Q$
3	$K_3 = K_2$ $S_3 = S_2$	$K_3 = K_2$ $S_3 = S_2$
4 I	$a = b$ $K_4 = K_3 - K\ (X + X^p)$ $K_4 \geqq O$ $S_4 = S_3 - A\ (X + X^p)$ $F_4 = X + X^p$	$b = a$ $K_4 = K_3 - K\ (X + X^p)$ $K_4 \geqq O$ $S_4 = S_3 - A\ (X + X^p)$ $F_4 = X + X^p$
5	$X^p = O$ $K_5 = K_4$ $S_5 = S_4$ $F_5 = F_4 = X$	$X^p \geqq O$ $K_5 = K_4 + VX^p$ $S_5 = S_4$ $F_5 = X$
6	$K_6 = K_5 - VX^p$ $S_6 = S_5$ $F_6 = F_5$	$K_6 = K_5$ $S_6 = S_5$ $F_5 = F_6$
7 II	$K_7 = K_6 - HY$ $K_7 \geqq O$ $S_7 = S_6 - BY$ $S_7 \geqq O$ $F_7 = F_6 - CY$ $F_7 \geqq O$ $P_7 = Y$	$K_7 = K_6 - (H + DC)\ Y$ $K_7 \geqq O$ $S_7 = S_6 - BY$ $S_7 \geqq O$ $F_7 = O$ $TX - CY \geqq$ $P_7 = Y$
MAX	$f = K_7 + US_7 + VF_7 + WY$	$f = K_7 + US_7 + (VX—PCY) + WY$

the combinatory operations, as there are two alternatives with each out of the six decisions. In fact, however, the number of the processes coming practically into consideration is pretty smaller, for some processes with their corresponding characteristics are economically useless and also absurd. The above mentioned number has to be reduced for such characteristics.

Let us first establish how many characteristics of type $(a_1, \ldots)$ are omitted; all such characteristics are $2^5 = 32$. Out of them the characteristics of type $(a_1, b_2, \ldots)$ are omitted, for in these casses we can purchase any quantities of the elements of production process, irrespective of the monetary means which are unlimited. Thus $2^4 = 16$ characteristics are omitted, and there remain $32 - 16 = 16$ characteristics of type $(a_1, a_2, \ldots)$. Among these characteristics, the characteristics of type $(\ldots, a_5, b_6 \ldots)$ are omitted, because of their absurdity; if, namely, the phase articles are not sold, we cannot use proceeds in the second production phase. There are $4 = 2^2$ such characteristics. Thus, there remain $16 - 4 = 12$ of them. And among these characteristics also the characteristics of type $(\ldots, b_5, b_6, \ldots)$ are omitted as the addition of the monetary means obtained from the sale of the phase articles to another phase has no sense, for the monetary means are unlimited; there are $2^2 = 4$ such characteristics. Thus, there remain altogether only $12 - 4 = 8$ usable characteristics of the type dealt with. Let us also establish how many characteristics of type $(b_1 \ldots)$ are omitted. Among them the characteristics of type $(\ldots, a_5, b_6 \ldots)$ are omitted, being absurd; for, if the phase articles are not sold, we cannot use the proceeds in the second phase; there are eight such characteristics. Thus, there remain $32 - 8 = 24$ characteristics of the type dealt with. Thus, there are altogether $8 + 24 =$ characteristics usable that come into consideration.

In the processes in which the phase articles are of such a nature that they are indivisible, e.g. machines, we should require that, the quantities of the phase articles produced be integers. Similarly, we should also require that the quantities of the finished articles be integers if the finished articles are indivisible which, e.g. occurs in the agricultural production in cattle-breeding. In such cases the integer linear programming should be used, which, however, is beyond our scope.

In analyzing the two-phase process we have, up to now, confined ourselves to the production processes. The established methodology, however, can be applied to the two-phase processes of other types.

IV. MULTI-PHASE PROCESSES

The general analysis we have dealt with for the two-phase production processes can be, without any special difficulties, generalized for processes occurring in more than two phases. It is clear that a larger

number of decisions occurs in multi- phase processes, where it is necessary to decide on one of the alternatives. Later on we shall deal with a very general production process occurring in n-phases, where n is any natural number. In this it is possible to see from the text which alternatives are chosen in the individual decisions. As in the two-phase process we assume that we start the multi-phase process with the monetary means k and with s various elements of the production process

$$Q_1, \ldots, Q_s \,.$$

There are only limited quantities available of these elements of the production process, namely successively at the most at

$$q_1, \ldots, q_s \,.$$

The elements of the production process are purchased successively at the prices

$$u_1, \ldots, u_s \,.$$

In fact, we purchase them before starting the production process successively at

$$z_1, \ldots, z_s \,.$$

After introducing matrices:

$$q = \begin{Vmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_s \end{Vmatrix} \qquad z = \begin{Vmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_s \end{Vmatrix} \qquad u = \begin{Vmatrix} u_1 & \ldots & u_s \end{Vmatrix}$$

for the monetary means the inequality is true:

$$U Z \geqq k, \qquad (0;\ 1)$$

and for the elements of the production process the inequality:

$$Z \leqq Q. \qquad (0;\ 2)$$

First phase. With the monetary means which have remained after the purchase of the elements of the production process we start the first production phase in which we produce the following phase articles of the first order:

$$F_1^1, \ \ldots \ F_{f_1}^1 \,.$$

The particulars of the production in the first phase are shown in Table 5.

Table 5. *First phase.*

		Phase articles of the first order $F^1_1 \;\cdots\; F^1_{f_1}$	Constraints
Monetary means K		$k^1_1 \;\cdots\; k^1_{f_1}$	Remainder
Elements of the production process	Q_1	$a^1_{11} \;\cdots\; a^1_{1f_1}$	z_1
	$\vdots$	$\vdots \qquad \vdots$	$\vdots$
	Q_s	$a^1_{s1} \;\cdots\; a^1_{sf_1}$	z_s
Quantities produced		$x^1_1 + x^{1p}_1 \;\cdots\; x^1_{f_1} + x^{1p}_{f_1}$	
Prices		$v^1_1 \;\cdots\; v^1_{f_1}$	

After introducing matrices:

$$K_1 = \| k^1_1 \;\cdots\; k^1_{f_1} \|$$

$$A^1 = \begin{Vmatrix} a^1_{11} & \cdots & a^1_{1f_1} \\ \vdots & & \vdots \\ a^1_{s1} & \cdots & a^1_{sf_1} \end{Vmatrix} \quad X^1 = \begin{Vmatrix} x^1_1 \\ \vdots \\ x^1_{f_1} \end{Vmatrix} \quad X^{1p} = \begin{Vmatrix} x^{1p}_1 \\ \vdots \\ x^{1p}_{f_1} \end{Vmatrix}$$

$$V^1 = \| v^1_1 \;\cdots\; v^1_{f_1} \|$$

we get the inequality for monetary means:

$$K^1 (X^1 + X^{1p}) \leqq K_1 \qquad (1\,;1)$$

and the inequality for the elements of the production process:

$$A^1 (X^1 + X^{1p}) \leqq Z. \qquad (1\,;2)$$

Second Phase. The second phase is started with the remaining monetary means increased by the proceeds from the phase articles of the first order, with the remaining elements of the production process and with the unsold phase articles of the first order. In the second phase the following articles of the second order are produced:

$$F^2_1, \;\ldots,\; F^2_{f_2}.$$

The particulars of the production in the second phase are shown in Table 6.

Table 6. *Second phase.*

		Phase articles of the second order $F^2_1 \;\cdots\; F^2_{f_2}$	Constraints
Monetary means K		$k^2_1 \;\cdots\; k^2_{f_2}$	Remainder K_2
Elements of the production process	Q_1	$a^2_{11} \;\cdots\; a^2_{f_2}$	Remainder S_2
	$\vdots$	$\vdots \qquad \vdots$	
	Q_s	$a^2_{s1} \;\cdots\; a^2_{sf_2}$	
Phase articles of the first order	F^1_1	$a^{12}_{11} \;\cdots\; a^{12}_{1f_1}$	x^1_1
	$\vdots$	$\vdots \qquad \vdots$	$\vdots$
	$a^1_{f_1}$	$a^{12}_{f_11} \;\cdots\; a^{12}_{f_1f_2}$	$x^1_{f_1}$
Quantities produced		$x^2_1 + x^{2p}_1 \;\cdots\; x^2_{f_2} + x^{2p}_{f_2}$	
Prices		$v^2_1 \;\cdots\; v^2_{f_2}$	

After introducing matrices:

$$K^2 = \| k^2_1 \;\cdots\; k^2_{f_2} \|$$

$$A^2 = \begin{Vmatrix} a^2_{11} & \cdots & a^2_{1f_2} \\ \vdots & & \vdots \\ a^2_{s1} & \cdots & a^2_{sf_2} \end{Vmatrix} \quad A^{12} = \begin{Vmatrix} a^{12}_{11} & \cdots & a^{12}_{1f_2} \\ \vdots & & \vdots \\ a^{12}_{f_11} & \cdots & a^{12}_{f_1f^2} \end{Vmatrix}$$

$$X^2 = \begin{Vmatrix} x^2_1 \\ \vdots \\ x^2_{f_2} \end{Vmatrix} \quad X^{2p} = \begin{Vmatrix} x^{2p}_1 \\ \vdots \\ x^{2p}_{f_2} \end{Vmatrix}$$

$$V^2 = \| v^2_1 \;\cdots\; v^2_{f_2} \|$$

we get the inequality for monetary means:

$$K^2 (X^2 + X^{2p}) \leqq K_2 + V^1 X^{1p} \qquad (2;1)$$

the inequality for the elements of the production process:

$$A^2 (X^2 + X^{2p}) \leqq S_2 \qquad (2;2)$$

and the inequality for the phase articles of the first order:

$$A^{12} (X^2 + X^{2p}) \leqq X^1 \qquad (2;3)$$

Thrid phase. We start the third phase with the monetary means increased by the proceeds from the sale of the phase articles of the second order, with the remaining elements of the production process, with the remaining phase articles of the first order, with the unsold phase articles of the second order. In the third phase, the phase articles of the third order are produced:

$$F_1^3, \; \ldots \; F_{f_3}^3 .$$

Similarly, as for the previous cases we draw up the Table for the first three phases introducing the corresponding matrices. After introducing these matrices we get inequality for the monetary means:

$$K^3(X^3 + X^{3p}) \leqq K_3 + V^2 X^{2p} , \qquad (3;\ 1)$$

the inequality for the elements of the production process:

$$A^3(X^3 + X^{3p}) \leqq S_3$$

the inequality for the phase articles of the first order:

$$A^{13}(X^3 + X^{3p}) \leqq F_{13}$$

and the inequality for the elements of the second order:

$$A^{23}(X^3 + X^{3p}) \leqq X^2.$$

n-Phase. This procedure is repeated from phase to phase until the last *n*-phase is reached. The last *n*-phase is started with the remaining monetary means, with the remaining elements of the production process, with the remaining phase articles of the first, second, . . . , (*n*- 2) orders and with the unsold phase articles of the order (*n*- 1). In the last phase the finished articles are produced:

$$P_1, \ldots , P_p .$$

The particulars of the production in the *n*-phase are shown in Table 7.

After introducing the corresponding matrices we get the following inequalities:

for the monetary means:

$$K^nY \leqq K_n + V^{n-1} X^{n-1,\,p} , \qquad (n,\ 1)$$

for the elements of the production process:

$$A^nY \leqq S_n, \qquad (n,\ 2)$$

Table 7. *n- Phase.*

		Finished articles P_1	. . .	P_p	Constraints
Monetary means	K	k_1^n	. . .	k_p^n	remainder K_n
Elements of the production process	Q_1	a_{11}^n	. . .	a_{1p}^n	remainder S_n
	.	.		.	
	Q_s	a_{s1}^n	. . .	a_{sp}^n	
Phase articles of the first order	F_1^1	a_{11}^{1n}	. . .	a_{1p}^{1n}	remainder F_{1n}
	.	.		.	
	$F_{f_1}^1$	$a_{f_11}^{1n}$	. . .	$a_{f_1p}^{1n}$	
. . . .		. . . .	. . . .	. . . . . .	. . . .
Phase articles of the (n—1) order	F_1^{n-1}	$a_{11}^{n-1,n}$	. . .	$a_{1p}^{n-1,n}$	x_1^{n-1}
	.	.		.	.
	$F_{f_{n-1}}^{n-1}$	$a_{f_{n-1}1}^{n-1,n}$	. . .	$a_{f_{n-1}p}^{n-1,n}$	$x_{f_{n-1}}^{n-1}$
Quantities produced		Y_1	. . .	Y_p	
Prices		W_1	. . .	W_p	

for the phase articles of the frst order:

$$A^{1n}Y \leqq F_{1n}. \qquad (n,\ 3)$$

: . . . . : . . . .

and for the phase articles of order (n — 1):

$$A^{n-1,\,n}Y \leqq X^{n-1}. \qquad (n,\ n + 2)$$

After choosing the optimality criterion we also determine the objeotive function:

$$f\ (Z,\ X^1,\ X^{1p},\ \ldots,\ X^{n-1},\ X^{n-1,\ p},\ Y) \qquad (n + 1)$$

and get the following formulation of the *n*-phase production process:

To determine the matrices:

$$Z,\ X^1,\ X^{1p},\ \ldots,\ X^{n-1},\ X^{n-1\ p},\ Y$$

which are not negative and which satisfy the enequalities:

(0; 1,2),

(1; 1,2),

(2; 1,2,3),

(3; 1,2,3,4),

. . . . . .

(n; 1,2, . . . , n + 2)

so that the objective funtion (n + 1) has the maximum.

V. CONCLUSION

In dealing with the multi-phase economic processes we have deliberately confined ourselves to the systematic analysis of these processes. For this reason special attention has been given to the formal construction of the corresponding mathematical models, thus somehow neglecting the economic problems which are, however, essential to all the practical problems of this kind. With our mathematical model we wanted to show the general algebra of the multi-phase economic processes without taking into consideration all the numerous possible circumstances that may play, even a decisive, part in such processes.

As usually, in constructing mathematical models the problems dealt with have been limited with various constraints and assumptions.

Particularly critical are the assumptions forming, the basis of the individual decisions possible. These assumptions have to be thoroghly checked in dealing with the concrete economic processes.

In decision D_2 concerning the source of the elements of the production process only two different alternatives were taken into account: that, when all the elements of the production process are directly available (alternative a_2) or, when they have to be purchased first (alternative b_2). These are, however, only extreme alternatives; there are many intermediate possibilities which have not been dealt with, and where there are some elements of the production process directly available, and the others when they have to be purchased first. A formal dealing with an intermediate possibility does not cause any special trouble as there is the question of combining both extreme alternatives in such a case.

Similar intermediate possibilities may occur in some other decisons as well.

Decision D_6 is especially problematic as there is to be decided what happens to the monetary means obtained from the sale of the phase articles. According to the alternative b_6, we can use them for financing another production phase; it is questionable, however, when this is possible altogether: namely, when is the economic decision-taking so quick, or when is the prouction process so slow that the realization of the phase articles can be engaged in the next phase.

In a complex multi-phase production there is in practice, a possiblity that the phase articles of the technologically later phases can be retrogradely engaged in the technologically previous phases. In spite of them, being quite frequent, such processes were not approached in our analysis.

The discussed multi-phase economic processes can be enriched by taking into consideration various other circumstances. Various factors, not yet taken into consideration, can be analyted, such as: various factors of production and especially market conditions. Enriching these models with an exhaustive market analysis could bear good results in the concrete research work.

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ПРОГРАММИРОВАНИЕ МНОГОФАЗНЫХ ПРОЦЕССОВ

Резюме

Каждый многофазный промышленный процесс можно разбить на последовательность отдельных фаз. При программировании многофазных процессов возникает вопрос, как оптимально запрограммировать отдельные фазы процесса относительно выбранной исходной функции.

Автор сперва систематически анализирует двухфазные процессы. Особое внимание он при этом посвящает конструированию соответствующих математических моделей. Принимая во внимание различные возможности развития двухфазного процесса, автор классифицирует процессы в различные типы, обозначаемые им специфичными характеристиками. Таким образом автор охватил 32 разных практически отмеченных типа двухфазных процессов.

Автор затем обобщил этот анализ на промышленные процессы, протекающие в произвольном количестве фаз.