BILINEAR PROGRAMMING

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I. FORMULATION OF THE GENERAL PROBLEM OF THE BILINEAR PROGRAMMING

Some problems of determining the optimum lead to a special type of non-linear problems with two independent series of variables the bilinear form of which is the objective function of the problem.

The general problem of non-linear programming with bilinear objective function can be formulated as follows:

The maximum of the bilinear objective function:

$$f = c_{11} x_1 y_1 + \ldots + c_{1\nu} x_1 y_\nu + \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ + c_{\mu 1} x_\mu y_1 + \ldots + c_{\mu\nu} x_\mu y_\nu$$

in which the variables.

$$x_1, \ldots, x_\mu; \; y_1, \ldots, y_\mu$$

satisfy the conditions of nonnegativity:

$$x_1 \geqq 0, \ldots, x_\mu \geqq 0; \; y_1 \geqq 0, \ldots, y_\nu \geqq 0$$

and the linear inequalities are:

$$a_{11} x_1 + \ldots + a_{1\mu} x_\mu \leqq p_1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{r1} x_1 + \ldots + a_{r\mu} x_\mu \leqq p_r;$$

$$b_{11} y_1 + \ldots + b_{1\nu} y_\nu \leqq q_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ b_{s1} y_1 + \ldots + b_{s\nu} y_\nu \leqq q_s.$$

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In it the numbers μ, ν, r and s are any natural numbers, and the coefficients, a, b, c, and q, represent any real numbers in all indexes.

The corresponding problem of the bilinear programming of the minimum bilinear function is formulated in a similar way, the formulation and treatment of the problem is left to the reader.

The formulated problem includes two independent series of variables which are not interconditioned. The variables of each series satisfy the condition of nonnegativity and each series has its group of linear inequations and equations. The objective function is the bilinear form of these variables.

We follow the procedure usually adopted when treating problems of linear programming. We first transform the inequalities into equations by introducing the corresponding slack variables. Upon introducing two series of slack variables:

$$x_{\mu+1}, x_m;\ y_{\nu+1}, \ldots, y_n$$

we arrive at the following problem of bilinear programming:

We have to determine the maximum of the bilinear objective function:

$$\begin{aligned} f = c_{11}x_1y_1 + \ldots + c_{1n}x_1y_n + \\ \cdots\cdots\cdots \\ + c_{m1}x_my_1 + \ldots + c_{mn}x_my_n \end{aligned}$$

such that the variables:

$$x_i\ (i = 1, 2, \ldots, m);\ y_j\ (j = 1, 2, \ldots, n)$$

satisfy the conditions of nonnegativity:

$$x_i \geqq 0;\ y_j \geqq 0$$

and the linear equations:

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1m}x_m = p_1, \\ \cdots\cdots\cdots \\ a_{r1}x_1 + \ldots + a_{rm}x_m = p_r; \end{aligned}$$

$$\begin{aligned} b_{11}y_1 + \ldots + b_{1n}y_n = q_1, \\ \cdots\cdots\cdots \\ b_{s1}y_1 + \ldots + b_{sn}y_n = q_s. \end{aligned}$$

In the formulated problem the following matrices occur:

$$A_{r \times m} = A_{r \times (\mu + r)} = \left\| \begin{matrix} a_{11} & \ldots & a_{1\mu} & 1 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{r1} & \ldots & a_{r\mu} & 0 & \ldots & 1 \end{matrix} \right\|$$

$$B_{s\times n} = B_{s\times(\nu+s)} = \left\| \begin{matrix} b_{11} & \dots & b_{1\nu} & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{s1} & \dots & b_{s\nu} & 0 & \dots & 1 \end{matrix} \right\|$$

$$C_{m\times n} = \left\| \begin{matrix} c_{11} & \dots & c_{1\nu} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{\mu 1} & \dots & c_{\mu\nu} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{matrix} \right\|$$

$$P = \left\| \begin{matrix} p_1 \\ \vdots \\ p_r \end{matrix} \right\| \qquad Q = \left\| \begin{matrix} q_1 \\ \vdots \\ q_s \end{matrix} \right\|$$

$$X = \left\| \begin{matrix} x_1 \\ \vdots \\ x_m \end{matrix} \right\| \qquad Y = \left\| \begin{matrix} y_1 \\ \vdots \\ y_n \end{matrix} \right\|$$

Having introduced these matrices we can reformulate the problem of bilinear programming in matrix form: To do this we have to determine the maximum of the bilinear objective function:

$$f = X^T C Y$$

under such that the vectors X and Y satisfy the conditions of nonegativity:

$$X \geqq 0,$$

$$Y \geqq 0$$

and the matrix equations:

$$A X = P,$$

$$B Y = Q.$$

The pair of vectors X and Y is called feasible solution of the formulated problem if vector X satisfies the condition of nonnegativity and the matrix equation $AX = P$, and if vector Y satisfies the condition of nonnegativity and the matrix equation $BY = Q$.

It follows from the theory of linear programming that the vectors X, which satisfy such conditions, form a convex polyhedron. Let us suppose that K_x is this convex polyhedron and that it has the following extreme points:

$$E_x^1,\ E_x^2,\ E_x^u.$$

Similarly, the vectors Y, which satisfy such conditions, form a convex polyhedron. Let us suppose that K_y is this convex polyhedron and that it has the following extreme points:

$$E_y^1.\ E_y^2, \ldots,\ E_y^v.$$

In what follows, the bilinear programming problem with two separate series of variables will be translated into a problem of bilinear programming in which the two series of variables will be joined so that we can deal with only one system of conditional equations, and with only one conditional matrix equation, respectively.

The vector space $P(X)$ is the set of all m-dimensional vectors, or points X; the convex polyhedron K_x being part of this vector space. The vector space $P(Y)$ is the set of all n-dimensional vectors, or points Y; the convex polyhedron K_y being part of this vector space.

The product of the vector spaces $P(X)$ and $P(Y)$ is the vector space $P(Z)$, which is the set of all $(m + n)$ dimensional vectors or points, respectively.

$$P(Z) = P(X)\ P(Y) = \left\| \begin{matrix} X \\ Y \end{matrix} \right\|$$

Each vector of the vector space $P(X)$ can be expressed by the corresponding vector Z of the vector space $P(Z)$ in the following way:

$$X = \left\| \begin{matrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ 0 & \ldots & 1 & 0 & \ldots & 0 \end{matrix} \right\| Z = \| I\ O \|\ Z.$$

Similarly, we can express each vector Y of the vector space $P(Y)$ by the corresponding vector Z of by the vector space $P(Z)$:

$$Y = \left\| \begin{matrix} 0 & \ldots & 0 & 1 & \ldots & 0 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{matrix} \right\| Z = \|\ O\quad I\ \|\ Z.$$

Let us also introduce matrices:

$$D = \begin{Vmatrix} a_{11} & \dots & a_{1m} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{r1} & \dots & a_{rm} & 0 & \dots & 0 \\ 0 & \dots & 0 & b_{11} & \dots & b_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & b_{s1} & \dots & b_{sn} \end{Vmatrix} = \begin{Vmatrix} A & O \\ O & B \end{Vmatrix}$$

$$R = \begin{Vmatrix} p_1 \\ \vdots \\ p_r \\ q_1 \\ \vdots \\ q_s \end{Vmatrix} = \begin{Vmatrix} P \\ Q \end{Vmatrix}$$

$$F_{(m+n)\times(m+n)} = \begin{Vmatrix} 0 & \dots & 0 & c_{11} & \dots & c_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & c_{m1} & \dots & c_{mn} \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{Vmatrix} = \begin{Vmatrix} O & C \\ O & O \end{Vmatrix}$$

Upon introducing these matrices the conditions of nonnegativity $X \geqq 0$ and $Y \geqq 0$ are joined into only one condition of nonnegativity $Z \geqq 0$; the matrix equations $YX = P$ and $BY = Q$ are joined into only one matrix equation $DZ = R$; the bilinear objective function gets the form: $f = Z^T F Z$. In this way, the primary problem with separated conditions has been translated into a problem in which the conditions are joined together.

The maximum of the bilinear objective function:

$$f = Z^T F Z$$

such that the vector Z satisfies the demand for nonnegativity:

$$Z \geqq 0$$

and the matrix equation:

$$D Z = R.$$

has been determined.

It follows from the theory of linear programming that the vectors Z, wich satisfies these conditions, form the convex polyhedron; this polyhedron can be marked with K_z.

II. THE PROPERTIES OF THE FEASIBLE SOLUTIONS

There are two theorems for the feasible solutions of both the primary problem with separated conditions and the new problem with joined conditions.

Theorem 1: The product of the vectors X and Y which represent the feasible solutions of the primary problem, is the feasible solution Z of the new problem in wich the conditions are joined fogether.

Proof: The vectors X and Y, which represent the feasible solution of the primary problem, correspond to the inequalities $X \geqq 0$ and $Y \geqq 0$; their product

$$Z = \begin{Vmatrix} X \\ Y \end{Vmatrix}$$

satisfies thus the inequality $Z \geqq 0$. Further, since vectors X and Y represent the feasible solution of the primary problem, they correspond to the equations $AX = P$ and $BY = Q$; their product thus satisfies the matrix equation:

$$\begin{Vmatrix} A & O \\ O & B \end{Vmatrix} \begin{Vmatrix} X \\ Y \end{Vmatrix} = \begin{Vmatrix} P \\ Q \end{Vmatrix}$$

It follows therefrom that their product Z corresponds to the equation $DZ = R$. The vector Z is thus a feasible solution of the new problem in which the conditions are joined and thereby the theorem has been proved.

It follows from this teorem that the product of any vector X of the convex polyhedron K_x and of any vector Y of the convex polyhedron K_y is a vector Z of the convex polyhedron K_z.

Theorem 2: Any feasible solution of the new problem, the one in which the conditions are joined, is the product of two vectors X and Y, which represent the feasible solution of the primary problem.

Proof: Since vector Z, feasible solution of the new problem, satisfies the inequality

$$Z = \begin{Vmatrix} X \\ Y \end{Vmatrix} \geqq 0;$$

vectors X and Y satisfy the condition of nonnegativity $X \geqq 0$ and $Y \geqq 0$. $DZ = R$, the matrix equation, is true:

$$\begin{Vmatrix} A & O \\ O & B \end{Vmatrix} \begin{Vmatrix} X \\ Y \end{Vmatrix} = \begin{Vmatrix} P \\ Q \end{Vmatrix}$$

It follows therefrom that the vectors X and Y satisfy equations $AX = P$ and $BY = Q$. The vectors X and Y thus represent the feasible solution of the primary problem, which had to be proved.

It follows from this theorem that each point Z of the convex polyhedron K_z is the product of a point X of the convex polyhedron K_x and of a point Y of the convex polyhedron K_y.

The following theorems are true for the vectors the X, Y and Z of the convex polyhedra K_x, K_y and K_z:

Theorem 1: The product of the two non-extreme points X and Y of the convex polyhedra K_x and K_y is the non-extreme point Z of the convex polyhedron K_z.

Proof: As the vector X is a vector of the convex polyhedron K_x, it can be expressed as a convex linear combination of the extreme vectors of the polyhedron K_x:

$$X = p_1 E_x^1 + \ldots + p_u E_x^u$$

where

$$0 \leqq p_1 \leqq 1, \ \ldots, \ 0 \leqq p_u \leqq 1;$$

$$p_1 + \ldots + p_u = 1.$$

As the vector X is not an extreme vector there are at least two positive coefficients among the coefficients p of the linear combination.

As the vector Y is a vector of the convex polyhedron Ky, it can be expressed as a convex linear combination of the extreme vectors of the polyhedron Ky, in this way:

$$Y = q_1 E_y^1 + \ldots + q_v E_y^v$$

where

$$0 \leqq q_1 \leqq 1, \ \ldots, \ 0 \leqq q_v \leqq 1;$$

$$q_1 + \ldots + q_v = 1.$$

As the vector Y is not an extreme vector there are at least two positive coefficients among the coefficients p of the linear combination.

Taking into consideration such an expression of the vectors X and Y we can put down the product of these two vectors in the following way:

$$\left\| \begin{matrix} X \\ Y \end{matrix} \right\| = \left\| \begin{matrix} p_1 E_x^1 + \ldots + p_u E_x^u \\ q_1 E_y^1 + \ldots + q_v E_u^v \end{matrix} \right\| =$$

$$= \sum_{i=1}^{i=u} \sum_{j=1}^{j=v} p_i q_j \left\| \begin{matrix} E_x^i \\ E_y^j \end{matrix} \right\| .$$

If

$$0 \leqq p_i q_j \leqq I$$

for each pair of the indexes i and j, and if

$$\sum_{i=1}^{i=u} \sum_{j=1}^{j=v} p_i q_j = \sum_{i=1}^{i=u} p_i \sum_{j=1}^{j=v} q_j = 1,$$

the product of the vectors X and Y is a convex linear combination of the vectors

$$\begin{Vmatrix} E_x^1 \\ E_y^1 \end{Vmatrix}, \ldots, \begin{Vmatrix} E_x^1 \\ E_y^v \end{Vmatrix}, \ldots, \begin{Vmatrix} E_x^u \\ E_y^1 \end{Vmatrix}, \ldots, \begin{Vmatrix} E_x^u \\ E_y^v \end{Vmatrix}$$

of the convex polyhedron K_z. Since at least four coefficients in the above expression are positive, the product of the vectors X and Y is a convex linear composition of at least four vectors of the convex polyhedron K_z, and therefore it is not an extreme vector.

Theorem 2: The product of a non-extreme point X of the polyhedron K_x and an extreme point E_y^i of the polyhedron K_y is a non-extreme point of the polyhedron K_z.

Proof: The non-extreme vector X of the polyhedron K_x can be expressed as a convex linear combination of the extreme points of the polyhedron:

$$X = p_1 E_x^1 + \ldots + p_u E_x^u ;$$

Since there are at least two positive coefficients p in this linear combination, the product can be expressed as follows:

$$\begin{Vmatrix} X \\ E_y^j \end{Vmatrix} = \begin{Vmatrix} p_1 E_x^1 + \ldots + p_u E_x^u \\ E_u^j \end{Vmatrix} = p_1 \begin{Vmatrix} E_x^1 \\ E_y^j \end{Vmatrix} + \ldots + p_u \begin{Vmatrix} E_x^u \\ E_y^j \end{Vmatrix}$$

The product is a convex linear combination of the vectors of the convex polyhedron K_z. Since at least two coefficients p of this combination are positive, the product is not an extreme point of the polyhedron K_z, which was to be proved.

Theorem 3 can be proved in a similar way: The product of an extreme point E_x^i of the convex polyhedron K_x and of a non-extreme point Y of the convex polyhedron K_y is a non-extreme point of the convex polyhedron K_z.

Theorem 4: The product of an extreme point E_x^i of the polyhedron K_x and of an extreme point E_y^j of the polyhedron K_y is an extreme point of the polyhedron K_z.

Proof: Let us suppose that the product

$$\begin{Vmatrix} E_x^i \\ E_y^i \end{Vmatrix}$$

is not an extreme vector of the polyhedron K_z. With this assumption there existed at least two vectors

$$Z^1 = \begin{Vmatrix} X^1 \\ Y^1 \end{Vmatrix}, \qquad Z^2 = \begin{Vmatrix} X^2 \\ Y^2 \end{Vmatrix}$$

such that the product would be the convex linear combination of these two vectors. Thus equation

$$\begin{Vmatrix} E_x^i \\ E_y^2 \end{Vmatrix} = p \begin{Vmatrix} X^1 \\ Y^1 \end{Vmatrix} + q \begin{Vmatrix} X^2 \\ Y^2 \end{Vmatrix}$$

would be completed under the conditions

$$0 < p < 1, \quad 0 < q < 1, \quad p + q = 1.$$

From the final equation follow the equations:

$$E_x^i = q\,X^1 + q\,X^2,$$

$$E_y^j = p\,Y^1 + q\,Y^2.$$

Thus vectors E_x^i and E_y^j are convex linear combinations of pairs of other vectors. This, however, is a contradiction, since both vectors are extreme. And thereby the theorem has been proved.

Theorem 5: Each extreme point E_z of the convex polyhedron K_z is the product of an extreme point of the vector E_x of the polyhedron K_x and of an extreme point E_y of the polyhedron K_y.

Proof: If the theorem were not true one of the three equations

a) $E_z = \begin{Vmatrix} X \\ E_y \end{Vmatrix}$, b) $E_z = \begin{Vmatrix} E_x \\ Y \end{Vmatrix}$, c) $E_z = \begin{Vmatrix} X \\ Y \end{Vmatrix}$

in which the vector X would be a non-extreme vector of the polyhedron K_x and the vector Y should exist as a non-extreme vector of the polyhedron K_y. It can be proved that none of the three possibilities mentioned can be true.

The non-feasibility of the first possibility can be proved as follows: Since X is a non-extreme vector of the polyhedron K_x, it can be expressed as a convex linear combination of the extreme points of the polyhedron:

$$X = p_1 E_x^1 + \ldots + p_u E_x^u;$$

in which at least two coefficients of the combination are positive. For this reason it could be possible to express the vector E_z through equations:

$$E_z = \left\| \begin{matrix} p_1 E_x^1 + \ldots + p_u E_x^u \\ E_y \end{matrix} \right\| = p_1 \left\| \begin{matrix} E_x^1 \\ E_y \end{matrix} \right\| + \ldots + p_u \left\| \begin{matrix} E_x^u \\ E_y \end{matrix} \right\|$$

in which at least two coefficients of the linear combination are positive. Vector E_z could be expressed as a convex linear combination of least two other vectors of the polyhedron K_z, and therefore, it would not be the extreme vector. The first possibility leads to contradiction and is rejected. In a similar way one can prove the impossibility of the two equations.

It follows from the proved theorems that the convex polyhedron K_z is the product of the convex polyhedra K_x and K_y; its extreme points are the possible products of each one extreme point of the polyhedron K_x and of each one extreme point of the polyhedron K_y. Thus the convex polyhedron has the following extreme points:

$$E_z^{11} = \left\| \begin{matrix} E_x^1 \\ E_y^1 \end{matrix} \right\|, \ldots, \quad E_z^{1v} = \left\| \begin{matrix} E_x^1 \\ E_y^v \end{matrix} \right\|$$

$$\vdots \qquad\qquad \vdots$$

$$E_z^{u1} = \left\| \begin{matrix} E_x^u \\ E_y^1 \end{matrix} \right\|, \ldots, \quad E_z^{uv} = \left\| \begin{matrix} E_x^u \\ E_y^v \end{matrix} \right\|$$

III. THE METHOD OF ITERATIONS

Seeking solution to problems of programming with bilinear objective function the iteration method starts with successive iterations, and permanently improves the solutions until the optimal feasible solution is attained.

Let us suppose that we start from the following first feasible solution of the problem:

$$Z^{11} = \left\| \begin{matrix} X^1 \\ Y^1 \end{matrix} \right\|.$$

This being a feasible solution, the conditions of nonnegativity are satisfied:

$$Z^{11} \geqq 0, \qquad \begin{matrix} X^1 \geqq 0, \\ Y^1 \geqq 0, \end{matrix}$$

and the matrix equations are completed:

$$D Z^{11} = R, \qquad \begin{matrix} A X^1 = P, \\ B Y^1 = Q; \end{matrix}$$

The value of the objective function for this feasible solution is:

$$f_1 = f(X^1, Y^1) = f(Z^{11}) = (Z^{11})^T \, F Z^{11}.$$

This first feasible solution is succeded by the second feasible solution in the following way:

In the convex polyhedron K_x we choose the vector X^1, which represents the first feasible solution. We insert this vector into the bilinear function $f(X, Y)$ thus producing a new function $f(X^1, Y)$, which is linear with regard to the elements of the vector Y. We calculate the maximum for this function which involves solution of the following linear programme:

$$\max f(X^1, Y); \quad Y \geqq 0 \;\&\; B\,Y = Q.$$

This linear programme is solved by a routine procedure. The optimal feasible solution Y^2 is an extreme point of the convex polyhedron K_y, and the corresponding maximum value of the objective function amounts to $f(X^1, Y^2)$ Since this value is optimal, the following equation is true:

$$f(X^1, Y^1) \leqq f(X^1, Y^2).$$

We continue the procedure in a similar way, always starting from the calculated vector Y^2. We insert the vector Y^2 into the bilinear function $f(X, Y)$ and thus produce the function $f(X, Y^2)$ which is linear with regard to the components of the variable vector X. We calculate the maximum for this function, by solving the following bilinear programme:

$$\max f(X, Y^2); \quad X \geqq 0 \;\&\; A\,X = P.$$

The optimal feasible solution X^2 of this linear programme is an extreme point of the convex polyhedron K_x and can be calculated according by routine technique the corresponding maximum value of the objective function amounts to $f(X^2, Y^2)$. Since this value is optimal, the following equation is true:

$$f(X^1, Y^2) \leqq f(X^2, Y^2).$$

We have concluded the first iteration, thereby attaining the second feasible solution:

$$Z^{22} = \begin{Vmatrix} X^2 \\ Y_2 \end{Vmatrix},$$

for which the objective function has the value:

$$f_2 = f(X^2, Y^2) = f(Z^{22}) = (Z^{22})^T \, F \, Z^{22}.$$

From the above inequalities it follows that the inequality

$$f(X^1, Y^1) \leqq f(X^2, Y^2),$$

is true and the second feasible solution is as good as or better than the first one.

As X^2 is the extreme point of the polyhedron K_x and Y^2 the extreme point of the polyhedron K_y, is point Z^{22}, which is a product of the points X^2 and, Y^2, the extreme point of the polyhedron K_z.

It follows, that the maximum of the bilinear functions is at an extreme point of the convex polyhedron K_z.

We can develop the second iteration in a similar way starting from the second feasible solution Z^{22}. We insert the vector X^2 into the bilinear function, thus attaining the linear programme:

$$\max f(X^2, Y); \quad Y \leqq 0 \;\&\; \mathrm{B}\,Y = Q.$$

The optimal feasible solution Y^3 of the bilinear function of this programme is the value $f(X^2, Y^3)$ which is not less than the previously calculated value $f(X^2, Y^2)$. The computed vector Y^3 is inserted into the bilinear function to yeald the linear programme:

$$\max f(X, Y^3) \quad X \leqq 0 \;\&\; A\,X = P.$$

The optimal feasible solution X^3 of the bilinear function of this programme is the value $f(X^3, Y^3)$ which is not less than the previously calculated value $f(X^2, Y^3)$. Having completed the second iteration we obtain the third feasible solution:

$$Z^{33} = \left\| \begin{matrix} X^3 \\ Y^3 \end{matrix} \right\|,$$

the bilinear function of which has the value:

$$f_3 = f(X^3, Y^3) = f(Z^{33}) = (Z^{33})^T\, F\, Z^{33}.$$

The calculated feasible solutions satisfy the inequalities:

$$f_1 \leqq f_2 \leqq f_3.$$

By repeating the iterative procedure we arrive at increasingly better feasible solutions all of which, including the second one, are extreme points of the convex polyhedron K_z. The procedure is completed as soon as a feasible solution is repeated. Since a known feasible solution can no more be improved the procedure is completed after n iterations when we arrive at

$$Z^{n,n} = Z^{n-1,\,n-1}.$$

As the number of extreme points of the convex polyhedron K_z is final, the described iterative procedure is, sooner or later, completed.

The maximum of the bilinear function is one extreme point of the convex polyhedrom K_z. This function can have a maximum at several different extreme points. We consider a number of different local maxima.

Each local maximum corresponds to one starting feasible solution Z^{11}, and can be calculated by the described iterations method. If the bilinear function has only one maximum, it can be calculated by the described method, irrespective of which feasible solution we start calculating. If, on the other hand, the problem allows for a number of local maxima the maximum, calculated by this method, depends upon the choice of initial feasible solution.

Using the method described we can calculate only one of the maxima of the bilinear function, which, however, is not necessarily the highest one. In spite of this deficiency the described method can be applied, to every given feasible solution which is not optimal. When dealing with problems of this kind we start from a feasible solution, known to be or intuitively felt as good and then improve it by the described method until the corresponding maximum is reached.

IV. PARCELLING OF POLYHEDRA

It follows from our discussion that the convex polyhedron K_x has the extreme points:

$$E_x^1,\ E_x^2,\ \ldots,\ E_x^u,$$

while the convex polyhedron K_y has the *extreme points:*

$$E_y^1,\ E_y^2,\ \ldots,\ E_y^v.$$

If we choose a certain point on the polyhedron X, we get the linear programme for Y:

$$LP(Y):\quad \max f(X,Y);\quad Y \geqq 0\ \&\ BY = Q,$$

with the optimal feasible solution, let us say, at the extreme point E_y^h of the polyhedron K_y.

$G_x(E_y^h)$ should be the set of all those points of the polyhedron K_x for which the corresponding linear programme $LP(Y)$ has the optimal feasible solution at the extreme point E_y^h of the polyhedron K_y.

We say that point X, for which the corresponding linear programme $LP(Y)$ has an optimal feasible solution at the point $E^h{}_y$, gravitates towards the extreme point E_y^h. In this case of the word gravitating $G_x(E_y^h)$ is the set of all those points X of the polyhedron K_x that gravitate towards the extreme point E_y^h of the polyhedron K_y. According to this terminology, the $G_x(E_y^h)$ will be from now on, referred to as the gravitational field of the extreme point E_y^h of the polyhedron K_y. A gravitational field can be defined as follows:

The gravitational field of the extreme point E_y^h of the polyhedron K_y is the set $G_x(E_y^h)$ of all those points X of the polyhedron K_x for which the corresponding linear programme:

$$LP(Y): \quad \max(X, Y); \quad Y \geqq O \;\&\; BY = Q$$

has an optimal feasible solution at the extreme point E_y^h.
To the extreme points:

$$E_y^1, E_y^2, \ldots, E_y^v$$

of polyhedron K_y, therefore, correspond, sucessively the following gravitational fields in polyhedron K_x:

$$G_x(E_y^1), \; G_x^2 (E_y), \ldots, G_x(E_y).$$

Similarly we determine the gravitational fields in the polyhedron K_y. If we choose the determined point Y in the polyhedron K_y, we obtain for X the following linear programme:

$$LP(X): \quad \max f(X, Y); \; X \geqq O \;\&\; AX = P,$$

which has the optimal feasible solution, let us say, at the extreme point E_x^g of the polyhedron K_x. $G_y(E_x^g)$ should be the set of all points of the polyhedron K_y, from which the corresponding linear programme $LP(X)$ has an optimal feasible solution at the extreme point E_x^g; the set of all these points Y of the polyhedron K_y forms the gravitational field of the extreme point E_x^g of the polyhedron K_x.
To the extreme points:

$$E_x^1, E_x^2, \ldots, E_x^u$$

of the polyhedron K_x correspond, successively the following gravitational fields in polyhedron K_y:

$$G_y (E_x^1), \; G_y (E_x^2), \ldots, G_y(E_x^u).$$

The following theorems are true for the gravitational fields:

Theorem 1: Each gravitational field is a convex set.

Proof: The theorem will be proved for any gravitational filed $G_x(E_y^h)$ which is a part of the polyhedron K_x; the proof for the gravitational field $G_y(E_x^g)$ which is part of the polyhedron K_y is an analogous one.

According to the definition of a gravitational field, all those points X of the polyhedron K_x belong to the set $G_x(E_y^h)$ for which the linear programme:

$$LP(Y): \quad \max f(X, Y); \; Y \geqq 0 \;\&\; BY = Q$$

has an optimal feasible solution at the extreme point E_y^h of the polyhedron K_y. In fact we have to prove the following theorem: *If*

$$X^1 \subset G_x(E_y^h)$$

and if

$$X^2 \subset G_x(E_y^h),$$

then

$$pX^1 + (1-p)\, X^2 \subset G_x(E_y^h);$$

since in a convex linear combination, p satisfies the inequality:

$$0 \leqq p \leqq 1.$$

The fact that the point X^1 is an element of the set $G_x(E_y^h)$ means that the linear programme:

$$LP(Y): \quad \max f(X^1, Y) = \max ((X^1)^T\, C\, Y); \quad Y \geqq 0 \;\&\; BY = Q$$

has an optimal feasible solution at the extreme point E_y^h, for which the objective function has the greatest value

$$(X^1)^T\, C\, E_y^h.$$

The fact that the point X^2 is an element of the set $G_x(E^h{}_y)$ means that the linear programme:

$$LP(Y): \quad \max f(X^2, Y) = \max ((X^2)^T\, C\, Y; \quad Y \geqq 0 \;\&\; BY = Q$$

has an optimal feasible solution at the point E_y^h For which the objective

$$(X^2)^T\, C\, E_y^h.$$

The fact that the convex linear combination

$$p\, X^1 + (1-p)\, X^2$$

of the points X^1 and X^2 is also the element of the set $G_x(E^h{}_y)$ means that the linear programme:

$$LP(Y): \quad \max f(pX^1 + (1-p)X^2, Y) = \max ((pX^1 + (1-p)X^2)^T\, CY;$$

$$Y \geqq 0 \;\&\; BY = Q$$

has an optimal feasible solution at the point E_y^h. for which the objective function has the greatest value

$$(pX^1 + (1-p)X^2)^T\, C\, E_y^h.$$

Let us suppose that the last linear programme has an optimal feasible solution at another point Y of the polyhedron K_y and not at the point E_y^h. In this case the inequality:

$$(pX^1 + (1-p)X^2)^T\, CY > (pX^1 + (1-p)X^2)^T\, C\, E_y^h$$

and, as well as the inequality:

$$p(X^1)^T C\, Y + (1-p)(X^2)^T\, C\, Y > p(X^1)^T C\, E_y^h + (1-p)(X^2)^T\, C\, E_y^h.$$

would be true. It follows that at least one of the inequalities:

$$(X^1)^T C\, Y > (X^1)^T C\, E_y^h.$$

$$(X^2)^T C\, Y > (X^2)^T C\, E_y^h.$$

should hold true. This is, however, impossible since the extreme point E_y^h is the optimal feasible solution of both the first and the second linear programme. And thereby the theorem has been proved.

The gravitational fields of the polyhedrons K_x, and K_y, respectively, do not cover each other, except perhaps at the points lying on the boundary of the gravitational fields. That is why no internal point of a gravitational field can be, at the same time, gravitational point of another gravitational field. This quality is the consequence of the theorem which we are going to formulate for two determined gravitational fields, but is evidently true for any pairs of gravitational fields.

Theorem 2: If

$$X^1 \subset G_x(E_y^1)$$

and if

$$X^2 \subset G_x(E_y^2),$$

where

$$E^1_y \neq E_y^2,$$

on the straight connecting line, between points X^1 and X^2 then there is one point at the utmost that

$$X \subset G_x(E_y^1)\ \&\ X \subset G_x(E_y^1).$$

is true; which means that among the convex linear combinations

$$X = pX^1 + (1-p)X^2, \quad 0 \leqq p \leqq 1$$

there is not more than one point X which at the same time belongs to the gravitational field $G_x(E_y)$ and to the gravitational field $G_x(E_y^2)$.

Proof: The relation

$$X^1 \subset G_x\ (E_y^1)$$

signifies that the linear programme

$$LP(Y): \quad \max\ (X^1)^T C\, Y; \quad Y \geqq 0\ \&\ B\, Y = Q$$

has the optimal feasible solution at the point E_y^1 and the maximum value of the objective function

$$(X^1)^T \, C \, E_y^1.$$

The relation

$$X^2 \subset G_x(E_y^2)$$

shows that the linear programme

$$LP(Y): \quad \max \, (X^2)^T \, C \, Y; \quad Y \geqq 0 \;\&\; B \, Y = Q$$

has the optimal feasible solution at the point E_y^2 and the maximum value of the objective function

$$(X^2)^T \, C \, E_y^2.$$

Let us take the convex linear combination

$$X = pX^1 + (1-p)X^2$$

of points X^1 and X^2. This linear combination lies on the straight connecting line between the points X^1 and X^2. We have to prove that there can be at most one point X, and at most one number p respectively, which satisfy:

$$X \subset G_x(E^1{}_y) \;\&\; X \subset G_x(E_y^2).$$

The relation

$$X \subset G_x(E_y^1)$$

shows that the linear programme

$$LP(Y): \quad \max \, (pX^1 + (1-p)X^2)^T \, C \, Y; \quad Y \geqq 0 \;\&\; B \, Y = Q$$

has an optimal solution at the point $E^1{}_x$ and the greatest value of the objective function

$$(pX^1 + (1-p)X^2)^T \, C \, E_y^1.$$

The relation

$$X \subset G_x(E_y^2)$$

means that the linear programme

$$LP(Y): \quad \max \, (pX^1 + (1-p)X^2)^T \, C \, Y; \quad Y \geqq 0 \;\&\; B \, Y = Q$$

has an optimal feasible solution at the point $E^2{}_y$ and the greatest value of the objective function

$$(pX^1 + (1-p)X^2)^T \, C \, E_y^2.$$

For the convex linear combinations

$$X = pX^1 + (1-p)X^2,$$

which belongs at the same time to the gravitaional field $G_x(E_y^1)$ and to the gravitational field $G_x(E_y^2)$ the last two greatest values of the objective function must be equal. It follows that convex linear combinations correspond to the equation:

$$(pX^1 + (1-p)X^2)^T\, C\, E_y^1 = (pX^1 + (1-p)X^2)^T\, C\, E_y^2,$$

from which we can calculate those values p that determine all points on the straight line between H^1 and H^2 that are common to both gravitational fields. With regard to the unknown p this equation can be either contradictory or linear , in no case, however, can it be of a higher degree. If the equation is linear, there exists only one solution, only one value p, and therefore only one point on the connecting line, belonging to both gravitational fields and delimitating them. If the equation is contradictory, there is no point on the connecting line that would be common to both gravitational fields. The straignt connecting line can in no case have more than one point that would be common to both gravitational fields. And thereby the theorem has been proved.

Theorem 3: The points on the boundary of two gravitational fields form a convex set.

Proof: If points X^1 and X^2 lie on the boundary of the gravitational fields $G_x(E_y^1)$ and $G_x(E_y^2)$ then their convex linear combination

$$X = pX^1 + (1-p)X^2$$

lies on the boundary of two gravitational fields.

If the point X^I lies on the boundary of the gravitational fields

$$G_x(E_y^1)\ \&\ G_x(E_y^2)$$

then

$$X^1 \subset G_x(E_y^1)\ \&\ X^1 \subseteq G_x(E_y^2)$$

is true and the corresponding linear programme

$$LP(Y):\quad \max\ (X^1)^T\, C\, Y;\quad Y \geqq 0\ \&\ B\, Y = Q$$

has two optimal feasible solutions

$$E_y^1\ \&\ E_y^2$$

for which the two greatest values of the objective function are equal:

$$(X^1)^T\, C\, E_y^I = (X^1)^T\, C\, E_y^2.$$

The same applies to point X^2. If the point X^2 lies on the border of both gravitational fields, then the corresponding linear programme has two optimal feasible solutions, for which both greatest values of the objective function are also equal. Now let us observe the convex linear combination

$$X = pX^1 + (1-p)X^2$$

of these two points.

Since X^1 is a point of the gravitational field $G_x(E_y^1)$ and since X^2 is a point of this gravitational field, the convex linear combination is a point of this gravitational field; for the gravitational field is a convex set.

Same applies to the second gravitational field. Since X^1 is a point of the gravitational field $G_x(E_y^1)$ and since X^2 is a point of this gravitational field each convex linear combination is a point of this gravitational field. And thereby the theorem has been proved.

Further, it can be proved that the linear programme, corresponding to the convex linear combination, at the points

$$E^1_y \quad \& \quad E_y^2$$

has equal greatest values of the objective function. The equation

$$(pX^1 + (1-p)X^2)^T C E_y^1 = (pX^1 + (1-p)X^2)^T C E_y^2,$$

is true, since the following equations are true:

$$(X^1)^T C E_y^1 = (X^1)^T C E_y^2,$$

$$(X^2)^T C E_y^1 = (X^2)^T C E_y^2.$$

It follows from the theorems proved that the polyhedron K_y is broken up into individual gravitational fields which are convex sets that do not mutually intersect in pairs except perhaps at boundary points. All boundary points at each of two gravitational fields form convex sets. That is why the boundary points forming the border between two gravitational fields, lie on a hyperplane; as each gravitational field is a section of the convex polyhedron K_x and of the semi-spaces determined by the corresponding hyperplanes, each gravitational field is also a convex polyhedron.

Thus the convex polyhedron K_x can be parcelled into the gravitational fields:

$$G_x(E_y^1),\ G_x(E_y^2),\ \ldots,\ G_x(E_y^r),$$

that are convex polyhedra and that do not cover each other, except perhaps at boundary points.

Similarly also the convex polyhedron K_y can be parcelled into the gravitational fields:

$$G_y(E_x^1),\ G_y(E_x^2),\ \ldots,\ G_y(E_x^u),$$

which are convex polyhedra that do not cover each other, except perhaps at boundary points.

It is clear that some of these gravitational fields can be empty sets.

Having parcelled the convex polyhedra K_x and K_y in the described manner, we can determine the local maxima, without any further calculation by mere determination of the gravitational field to which certain point belongs. The whole procedure of determining is based upon the previously treated method of itërations.

The starting-point for the iterations is the first feasible solution:

$$Z^{11} = \begin{Vmatrix} X^1 \\ Y^1 \end{Vmatrix},$$

to which the value of the objective function

$$f(Z^{11}) = f(X^1, Y^1).$$

corresponds.

X^1 is a point of the convex polyhedron. K_x, for which the gravitational field is detemined as say $G_x(Y^2)$. The point Y^2 say belongs to the convex polyhedron K_y; its gravitational field is say $G_y(X^2)$. In this way we have arrived at the second feasible solution:

$$Z^{22} = \begin{Vmatrix} X^2 \\ Y^2 \end{Vmatrix},$$

to which corresponds the objective function

$$f(Z^{22}) = f(X^2, Y^2).$$

We continue the procedure. We determine the gravitational field to which X^2 belongs; suppose $G_x(Y^3)$. Now we determine to which field Y^3 belongs; suppose it belongs to the field $G_y(X^3)$. Thus we have arrived at the third feasible solution:

$$Z^{33} = \begin{Vmatrix} X^3 \\ Y^3 \end{Vmatrix},$$

to which corresponds the value of the objective function

$$f(Z^{33}) = f(X^3, Y^3).$$

The iterations are repeated as long as a feasible solution is repeated

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BILINEARNO PROGRAMIRANJE

Alojzij VADNAL

R e z i m e

Članak tretira teoriju specijalnog tipa nelinearnog programiranja sa dve nezavisne serije promenljivih čiji bilinearni oblik predstavlja funkciju cilja. Problem je formulisan na sledeći način:

Vektori x *i* y *treba da se odrede tako da zadovolje uslove nenegativnosti:* $x \leqq 0$, $y \leqq 0$ *i nejednačine matrica* $Ax \leqq P$, $By \leqq Q$ *tako da bilinearna funkcija cilja dostiže svoj maksimum.*

Pored ove teorije u članku su dati algoritmi za određivanje lokalnih maksimuma funkcije cilja.