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Matrix Theory Application in the Bootstrapping Method for the Term Structure of Interest Rates*

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ABSTRACT *– This article focuses on the term structure of interest rates analysis in the form of a yield curve. The yield curve is a basic instrument for understanding the relationship between the price of money and the maturity of a financial instrument. It has the same relevance for all economic subjects in the form of a basic value determination. The term structure analysis can be used in different economic categories like financial management, portfolio management, actuary science, company valuation, management of firm value, financial risk management, etc. Such as basic method applied in the yield curve construction is the bootstrapping method. Unfortunately, there is great computing severity related to this method. Fortunately, however, the application of matrix theory helps us to solve this issue very well*

KEY WORDS: *yield curve, interest rate term structure analysis, bootstrapping method*

Introduction

The term structure of interest rates, also known as yield curve, is a static function that relates the term to maturity to the yield to maturity for a sample of bonds at a given point in time. Thus, it represents a cross section of yields for a category of bonds that are comparable in all respects but maturity.

Specifically, the quality of the issues should be constant, and ideally you should have issues with similar coupons and call features within a single industry category. So you can construct different yield curves for Treasuries, government agencies, prime grade municipals, AAA utilities, and so on. The accuracy of the yield curve will just depend on the comparability of the bonds in the sample.

In generally yield curves do not have the same shape at different points in time, thus term structure shape changes depends the time and depends the term to maturity. Depends on different term to maturity we knows the rising yield curve, declining yield curve, the flat yield curve or the humped yield curve. The rising yield curve is the most common and tends to prevail when interest rates are at low or modest levels. The declining yield curve tends to occur when rates are relatively high. The flat yield curve rarely exists for any period of time. The humped yield curve prevails when extremely high rates are expected to decline to more normal levels. Note that the slope of the yield curve tends to level off in general after 15 years.

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Different shape of the term structure or yield curve explains three major yield curve theories - the expectations hypothesis, the liquidity preference hypothesis, and the segmented market hypothesis [John Y. Campbell (1986)], [John C. Cox, Jonathan E. Ingersoll, Stephen A. Ross (1985)].

Theoretical spot rate curve estimation using bootstrapping method

The yield on a zero coupon bond for a given maturity is the spot rate for the maturity. Specifically, the spot rate is defined as the discount rate for a cash flow at a specific maturity. At that time, we used the rates on a series of zero coupon government bonds created by stripping coupon government bonds.

In this case, we will construct a theoretical spot rate curve from the observable yield curve that is based on the existing yields of Treasury bills and the most recent Treasury coupon securities [Sanjay K. Nawalkha and Donald R. Chambers (1999)]. One might expect the theoretical spot rate curve and the spot rate curve derived from the stripped zero coupon bonds to be the same.

The fact is, while they are close, they will not be exactly the same because the stripped zero coupon bonds will not be as liquid as the on-the-run issues. In addition, there are instances where institutions will have a strong desire for a particular spot maturity and this preference will distort the term structure relationship. Therefore, while it is possible to use the stripped zero coupon curve for a general indication, if you are going to use the spot rates for significant valuation, you would want to use the theoretical spot rate curve.

The process of creating a theoretical spot rate curve from coupon securities is called bootstrapping wherein it is assumed that the value of the Treasury coupon security should be equal to the value of the package of zero coupon securities that duplicates the coupon bond's cash flow.

The theoretical price of zero coupon bond (P) should equal to

$$P = F_t \cdot e^{-y \cdot t} = F_t \cdot d(t), \tag{1}$$

where $d(t) = e^{-y \cdot t}$.

Consider set of $\mathbf{N}$ coupon securities with semi annual coupon payment and continuous interest running. Then the 0,5 year coupon security should equal to 0,5 year zero coupon security containing the face value and 0,5 year coupon payment . The general relation for the theoretical price of this bond [Vincent Šoltés a Michal Šoltés (2007)] is

$$P = \sum_{j=1}^{N} C_{t_i} \cdot e^{-y \cdot t_i} + F_{t_N} \cdot e^{-y \cdot t_N} \tag{2}$$

The theoretical price of 0,5 year coupon bond is then

$$P(0{,}5) = \left(C_{0,5} + F_{0,5}\right) \cdot e^{-y(0,5) \cdot 0,5} \tag{3}$$

where $F_{0,5}$ is the face value, $C_{0,5}$ is 0,5 year coupon payment paid at the maturity of bond, and $y(0{,}5)$ is the theoretical spot rate of 0,5 year zero coupon bond or 0,5 year zero coupon yield. After simplifying

$$y(0{,}5) = \frac{1}{0{,}5} \cdot ln\left[\frac{C_{0,5} + F_{0,5}}{P(0{,}5)}\right] \tag{4}$$

To calculate the 1 year zero coupon we can use the price of a 1 year coupon bond and the theoretical spot rate of 0,5 year zero coupon bond, i.e.

$$P(1) = C_1 \cdot e^{-y(0,5)\cdot 0,5} + (C_1 + F_1) \cdot e^{-y(1)} \quad (5)$$

where F_1 is the face value, C_1 is the 0,5 year coupon paid at the end of 0,5 year and 1 year, and $y(1)$ is the 1 year zero coupon yield. After simplifying we get

$$y(1) = ln\left[\frac{C_1+F_1}{P(1)-\frac{C_1}{e^{y(0,5)\cdot 0,5}}}\right] \quad (6)$$

Using 0,5 zero coupon yield form equation (4) in the equation (6) we compute the theoretical spot rate of 1 year zero coupon bond.

Following the same approach the theoretical spot rates (or zero coupon yields) fall of the N maturities are calculated using the zero coupon yields of the previous maturities. In general about 15 maturities are sufficient to produce the whole yield curve of zero coupon bonds.

Using the matrix theory in the bootstrapping method

To obtain a direct solution we can apply matrix theory there. Let we consider N bonds maturing at dates $t_1, t_2, t_3, \cdots, t_N$, and let CF_{it} be the total cash flow payments of ith bond (for $i = 1, 2, 3, \cdots, N$) on the date t (for $t = t_1, t_2, t_3, \cdots, t_N$). Then the prices of N bonds equal to

$$\begin{pmatrix} P(t_1) \\ P(t_2) \\ \vdots \\ P(t_N) \end{pmatrix} = \begin{pmatrix} CF_{1t_1} & 0 & \cdots & 0 \\ CF_{2t_1} & CF_{2t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ CF_{Nt_1} & CF_{Nt_2} & \cdots & CF_{Nt_N} \end{pmatrix} \begin{pmatrix} d(t_1) \\ d(t_2) \\ \vdots \\ d(t_N) \end{pmatrix} \quad (7)$$

Generally described matrix P equals to matrix product AD of matrix A and matrix D, then

$$P = A \cdot D \quad (8)$$

After simplifying to the matrix D

$$D = A^{-1} \cdot P,$$

where the matrix A^{-1} presents the square matrix to the matrix A. To determine the inverse, we calculate using determinant of A

$$D = \frac{1}{|A|} \cdot [CF_{it}]_{n\times n} \cdot P \text{, where} \quad (9)$$

$$CF_{it} = (-1)^{i+t} |A_{ti}|,$$

and where A_{ti} represents square matrix of $(n-1)\times(n-1)$ type from the matrix A.

As you could see in (7), the upper triangle of the cash flow matrix A on the right side has zero values. By multiplying both sides of equation (7) by the inverse matrix of the cash flow, we get the discount functions corresponding to maturities, then

$$\begin{pmatrix} d(t_1) \\ d(t_2) \\ \vdots \\ d(t_N) \end{pmatrix} = \begin{pmatrix} CF_{1t_1} & 0 & \cdots & 0 \\ CF_{2t_1} & CF_{2t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ CF_{Nt_1} & CF_{Nt_2} & \cdots & CF_{Nt_N} \end{pmatrix}^{-1} \begin{pmatrix} P(t_1) \\ P(t_2) \\ \vdots \\ P(t_N) \end{pmatrix} \quad (10)$$

Note that this solution requires that the number of bonds equal the number of cash flow maturity dates.

Matrix theory application in the bootstrapping method

To demonstrate the bootstrapping method using the matrix approach we assume 10 coupon bonds and their parameters given in the table below. For simplicity we assume all bonds make annual coupon payments. The face value of bonds is 100 €.

Bond	Price (in €)	Maturity(in years)	Coupon rate (in per cent p.a.)
1	96,60	1	2
2	93,71	2	2,5
3	91,56	3	3
4	90,24	4	3,5
5	89,74	5	4
6	90,04	6	4,5
7	91,09	7	5
8	92,82	8	5,5
9	95,19	9	6
10	98,14	10	6,5

Using the matrix theory in the bootstrapping method given by equation (10) we get the discount functions corresponding to maturities, then

$$\begin{pmatrix} d(1)\\ d(2)\\ d(3)\\ d(4)\\ d(5)\\ d(6)\\ d(7)\\ d(8)\\ d(9)\\ d(10) \end{pmatrix} = \begin{pmatrix} 102 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 2{,}5 & 102{,}5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 3 & 3 & 103 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 3{,}5 & 3{,}5 & 3{,}5 & 103{,}5 & 3{,}5 & 0 & 0 & 0 & 0 & 0\\ 4 & 4 & 4 & 4 & 104 & 0 & 0 & 0 & 0 & 0\\ 4{,}5 & 4{,}5 & 4{,}5 & 4{,}5 & 4{,}5 & 104{,}5 & 0 & 0 & 0 & 0\\ 5 & 5 & 5 & 5 & 5 & 5 & 105 & 0 & 0 & 0\\ 5{,}5 & 5{,}5 & 5{,}5 & 5{,}5 & 5{,}5 & 5{,}5 & 5{,}5 & 105{,}5 & 0 & 0\\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 106 & 0\\ 6{,}5 & 6{,}5 & 6{,}5 & 6{,}5 & 6{,}5 & 6{,}5 & 6{,}5 & 6{,}5 & 6{,}5 & 106{,}5 \end{pmatrix}^{-1} \begin{pmatrix} 96{,}60\\ 93{,}71\\ 91{,}56\\ 90{,}24\\ 89{,}74\\ 90{,}04\\ 91{,}09\\ 92{,}82\\ 95{,}19\\ 98{,}14 \end{pmatrix}$$

Multiplication of these two matrices gives the solution

$$\begin{pmatrix} d(1)\\ d(2)\\ d(3)\\ d(4)\\ d(5)\\ d(6)\\ d(7)\\ d(8)\\ d(9)\\ d(10) \end{pmatrix} = \begin{pmatrix} 0{,}947059\\ 0{,}891145\\ 0{,}835392\\ 0{,}781473\\ 0{,}729997\\ 0{,}681409\\ 0{,}635787\\ 0{,}592963\\ 0{,}553006\\ 0{,}515742 \end{pmatrix}$$

The theoretical spot rates are obtained from the corresponding discount functions derived from equation (1) or

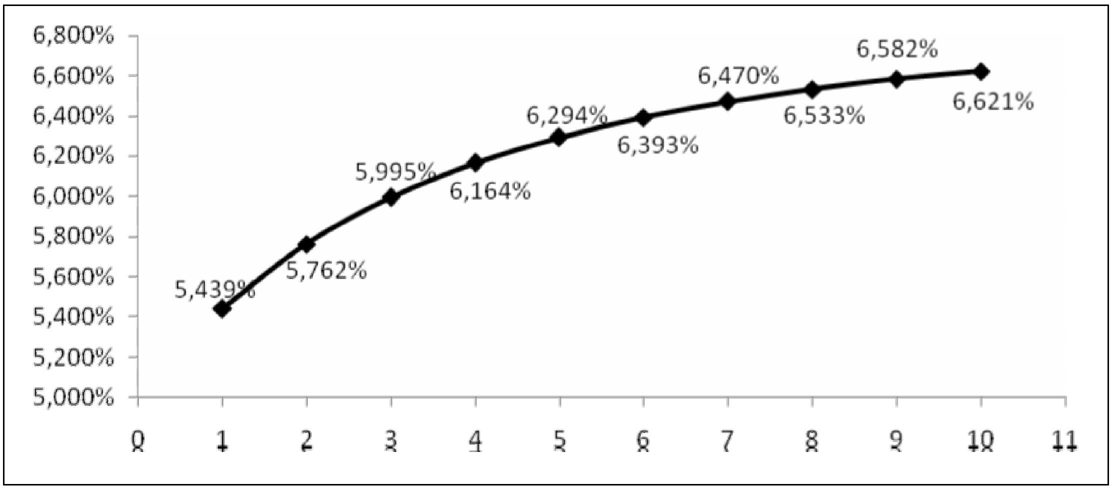
$$y(t) = \frac{-\ln d(t)}{t}$$

The particular zero coupon yields are depictured in Table 1. and Figure 1.

Table 1. Particular zero coupon yields (in per cent p. a.)

t	$y(t)$
1	5,439%
2	5,762%
3	5,995%
4	6,164%
5	6,294%
6	6,393%
7	6,470%
8	6,533%
9	6,582%
10	6,621%

Figure 1. Theoretical spot rate curve



Conclusion

The yield curve (or the term structure of interest rates) shows the relationship between the yields on a set of comparable bonds and the term to maturity. Based upon this yield curve, it is possible to derive a theoretical spot rate curve. Such as basic method applied for derivation of the yield curve (or several theoretical spot rates) is the bootstrapping method. Unfortunately, there is great computing severity related to this method. Fortunately, however, the application of matrix theory helps us to solve this issue very well, as we suggested in this paper. In turn, these spot rates can be used to value bonds using an individual spot rate for each cash flow. This valuation approach is becoming more useful in a world where bonds have very different cash flows. In addition, these spot rates imply investor expectations about future rates referred to as forward rates.

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