ECONOMIC ANALYSIS AND WORKERS' MANAGEMENT, 3, XXI (1987), 289—297

# SRAFFA SYSTEMATIZED AND MARX VINDICATED

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In order to analyse distribution relations independently of prices, Sraffa invented standard commodity; in order to analyse capitalist exploitation independently of market prices, Marx encountered the transformation problem.

The two procedures apparently have nothing in common. And yet they have. The unifying concept is the concept of the uniform organic composition of capital. Besides, Sraffa's goal may be achieved in three different ways. And Marx's transformation problem is not necessarily insoluble: his two conditions — the sum of values is equal to the sum of prices and the sum of surplus values is equal to the sum of profits — can be simultaneously satisfied.

The two problems are related in the following way. By exploiting the duality property of price-quantity relationships it is possible to derive a Sraffian linear relation between profits and wages in two different ways: by finding the left eigenvector of A (prices) and by finding the right eigenvector of A (quantities)[1]. If prices are formed by adding margins proportional to wages and equal to material costs, Marx's transformation problem is solved in a natural way. Both Sraffa's and Marx's problems reduce to imposing a uniform organic composition of capital on the system of prices which is a far less restrictive condition then generally believed.

The model used involves circulating capital and single product industries. The duality of price-quantity relationships can conveniently be decomposed into two problems: the primal problem consisting in the derivation of prices, and its dual consisting in deriving consistent quantities.

## THE PRIMAL PROBLEM

Consider an economy using only circulating capital, practicing simple reproduction and not experiencing technical change. In spite

[1] That has already been established by G. Abraham—Frois, E. Berrebi, (1975, pp. 275, 311 and passim) and L. Pasinetti (1977, pp. 100, 112—114).

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of the last two characteristics, the rate of profits is positive, $\pi > 0$, since the economy is run on capitalist principles. These are the assumptions common to both Marx (1894) and Sraffa (1960). Let A be the usual matrix of input coefficients, p the row vector of prices, w the wage rate and $\lambda$ the row vector of labour coefficients. Then the familiar price equations are given by

$$pA(1+\pi) + w\lambda = p \tag{1}$$

The system is nonlinear which has quite interesting economic consequences. First, if we knew prices, *and* prices were unique, wages would be a linear function of profits:

$$w = \frac{p(I-A)1}{\lambda 1} - \pi \frac{pA1}{\lambda 1}$$

where 1 is the summation vector. The set of unique prices will be denoted as p*. Second, prices must be positive and that, under certain conditions, makes them unique. Third, it will turn out that, as a consequence, organic composition of capital is quite an important analytical concept.

Equation (1) can be transformed in three different ways. Let me first follow Sraffa and set $w = 0$ to get a system with the maximum rate of profit $\pi$

$$p^*A(1+\pi) = p^*, \quad w = 0 \tag{2}$$

Since A is nonnegative and indecomposable, it will have a unique dominant eigenvalue $\frac{1}{1+\pi}$ with which the only positive eigenvector of prices, $p > 0$, is associated. $\pi$ is also a uniform ratio of net products and costs (capitals) in all industries, as it follows directly from (2)

$$p^*(I-A) = \pi p^* A \tag{2a}$$

The second possibility is to consider a non-capitalist variant and set profits equal to zero, $\pi = 0$, which will lead to maximum wages W

$$p^* A + W\lambda = p^*, \quad \pi = 0 \tag{3}$$

This also implies that prices will be labour time prices. For $W = 1$, prices will be exact labour prices (Marxian values). In Marxian terms $p^*A = C$ is constant capital and $W\lambda = V$ is variable capital. If we postulate uniform organic composition of capital in the system

$$p^* A = \omega W \lambda \tag{4}$$

there is again proportionality between costs and net outputs, the factor of proportionality beeing $\omega = \frac{1}{\pi}$.

The equation (1) is now transformed into

$$p^* A \left( 1 + \frac{1}{\omega} \right) = p^* \tag{5}$$

Since A is the same as before, prices as its eigenvector, do not change, and so it follows that

$$\pi = \frac{1}{\omega} \tag{6}$$

maximum profit in Sraffa is equal to the reciprocal of the uniform organic composition of capital in Marx.

It has long been known that uniform organic composition of capital generates a linear relation between w and $\pi$.[2] Equate unit net products in the three equations (1), (2) and (3)

$$p^* A \pi + w \lambda = p^* A \pi = W \lambda \tag{7}$$

Eliminate $\lambda$ to get

$$\pi = \pi \left( 1 - \frac{w}{W} \right) \tag{8}$$

If we normalize $W = 1$, (8) appears to be Sraffa's equation $\pi = \pi$ $(1 - w)$ with w representing the proportion of the maximum wage rate W.

## THE DUAL PROBLEM

While the primal solution deals with columns, the dual solution will deal with the rows of the same technological matrix A. Instead of constructing prices in an appropriate way, we shall modify the quantities. We start from quantity equations

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[2] Pasinetti noticed that maximum $\Pi$ generates the same effect and he also added the case when wages are expressed in terms of standard commodity (1977, p. 115). The reversed implication was left unnoticed. Namely, insofar as a unique set of prices generates a linear relationship between profits and wages, relative prices remain constant. Distribution relations independent of prices imply prices independent of distribution.

$$AX + x = X \tag{9}$$

Since A is productive, any vector of final outputs x can be produced. We use the available degrees of freedom to make final outputs proportionate to intermediate outputs (and, consequently, to total outputs)

$$AX(1 + R) = X \tag{10}$$

It turns out that $\frac{1}{1 + R}$ is an eigenvalue of the same matrix A, and X is the associated eigenvector. The only positive X corresponds to the maximum eigenvalue and so $R = \pi$. Of course, we knew that beforehand because value added and value of final output must be the same and so

$$R = \frac{x_i}{X_i - x_i} = \frac{p_i^* x_i}{p_i^* (X_i - x_i)} = \frac{\Sigma p_i^* x_i}{\Sigma p_i^* (X_i - x_i)} =$$

$$= \frac{p^* x}{p^* A X} = \frac{p^* (I - A) X}{p^* A X} = \pi = \frac{1}{\omega} \tag{11}$$

The standard ratio R is equal to the notional maximum profit rate $\pi$ and both are equal to the reciprocal of the uniform organic composition of capital.

If the system is composed of industries in such a way that total and intermediate outputs are proportional (which also makes values of outputs proportional to values of inputs), all industries will have the same organic composition of capital which will render the relation between $\pi$ and w linear. Such a system Sraffa called the standard system. Its activity levels X are determined as eigenvectors of the technology matrix A. Multipliers X are determined up to the scalar multiple and the available degree of freedom may be used for other analytical purposes. The standard ratio R is independent of prices and the same ratio $\pi$ does not change if the distribution between wages and profit changes, i.e., in the standard system the distribution is independent of prices.

## UNIFORM GROWTH

Suppose wages remain at subsistence level all the time. This was the standard assumption of classical economists. With fixed wages and unchanged preferences, wage goods can be included in the reproduction matrix A. As a result, profits appear as a sole surplus or net product, which was also a fairly usual assumption of our classical forebears. Add the modern Golden Age assumption that all profits are invested and you get the von Neumann growth model. If this economy is to grow at the maximal rate, there must be no waste. In other words, the economy must have an appropriate physical structure. The structure is rather simple: all final outputs, which will be fully invested, must bear the same relation R to intermediate outputs in the same industries. As a result the economy grows at the rate $\tilde{\pi} = \tilde{R}$ and the foregoing analysis applies.[3] But there is a difference. The need to have an appropriate physical structure makes uniform organic composition of capital a necessary but no longer a sufficient condition. The standard ration $\tilde{R}^*$ is now both necessary and sufficient. (Note that $\tilde{R} < R$ corresponds to augmented $\tilde{A} > A$ which includes subsistence consumption).

Yet even algebra does not require such unscrupulous capitalist exploitation which would repress wages to a bare existential minimum. In fact we may wish to maximize wages, given technology. If labour force grows at the rate $r$, this is obviously the maximum possible rate of growth of the system. If wages are expressed in the value of the wage goods y

$$w = py \tag{12}$$

the usual price equation will be tranformed into

$$p\,A\,(1 + r) + p\,y\,\lambda = p \tag{13}$$

which is algebraically similar to our starting equation. Since Perron-Frobenius theorems will again do their job, we can derive a linear relation between the rate of growth and per capita consumption

$$r = R\left(1 - \frac{p\,y}{p\,Y}\right) \tag{14}$$

where Y is a vector of final outputs per capita. As long as the structure of wage goods does not change, y is proportional to Y, and so further simplification is possible. Suppose $y_s$ is the subsistence collection of wage goods. Normalize

$$py_s = c_s = 1 \tag{15}$$

to obtain a subsistence basket. Now r — py relation is transformed into

$$r = R\left(1 - \frac{c}{C}\right) \tag{16}$$

[3] Cf. Pasinetti (1977, 204).

where C is the number of maximally possible subsistence baskets and c is the number of subsistence baskets actually consumed. Obviously, C is also a measure of the productivity of the system.

When studying consumption-growth relationships it is more natural to work with the dual solution. Thus, instead of using the columns, I shall now use the rows of A. Define final output coefficient $a_i$ as final output per unit of total output $X_i$ in the same industry $a_i = x_i/X_i$. Then the summation along the rows gives the output equation

$$AX\,(1 + r) + fa = X \tag{17}$$

where r replaces $\pi$, *a* takes the place of the labour coefficients $\lambda$ and f replaces wage rate. In fact vector a gives the structure of consumer demand and f is the level of consumption. Consumption at the unit level may be defined as f = 1. At the same time this will be the subsistence level if we normalize $fa = a_s$, where vector $a_s$ is the subsistence consumption of the employed labour force L. We note in passing that $a_s = Ly_s$, y being defined in (22). It is obvious that the rate of growth *r* can be expressed as a linear function of the level of consumption if we apply appropriate multipliers. Interpret the multipliers as prices and write

$$p\,A\,X\,(1 + r) + f\,p\,a = p\,X$$

Derive the same two equations as before (7)

$$r\,p\,A\,X + f\,p\,a = R\,p\,A\,X = F\,p\,a \tag{18}$$

where F is the maximum level of consumption at which total net output is consumed. Eliminate pa and the desired equation emerges

$$r = R\left(1 - \frac{f}{F}\right) \tag{18a}$$

It was shown above that the necessary and sufficient condition for the linearity of the primal problem was that the organic composition of capital was the same in all industries

$$pA = \omega\,(w\,\lambda)$$

Substitute $p\,A\,\pi = W\,\lambda$ from (7) to get

$$\omega = \frac{W}{\pi\,w} \tag{19}$$

If we construct prices in such a way that the value of final output is proportional to the value of intermediate output

$$p\,A\,X = \omega' f\,p\,a$$

and apply the same substitution RpAX = Fpa from (18) the result is a sort of a dual of organic composition

$$\omega' = \frac{F}{Rf} \tag{20}$$

As already mentioned, not every interpretation described is equally useful for all purposes. In general, column interpretations with the associated $\omega$ are more natural for a stationary economy and row interpretations with $\omega'$ seem to be better suited for the analysis of uniform growth.

## MARX VINDICATED

We now turn our attention to Marx. Similarly to other classical economists — and unlike Sraffa — he considers wages as being advanced, i.e., as being invested in variable capital. Consequently, the price equation will assume the following look

$$(pA + w\lambda)\,(1 + \pi) = p \tag{21}$$

Fortunately, the change has no important algebraic consequences. Impose labour prices ($\pi = 0$) and uniform organic composition of capital ($pA = \omega\,(w\lambda)$), and the familiar relation makes its appearance.

$$p^*A = \frac{\omega}{1 + \omega}\,p^* \tag{22}$$

Since income distribution does not affect prices $p^*$, the transformation of values into prices is easily carried out. In fact, prices and values are identical. In order to show that, it suffices to form a parallel systems of values

$$vA + w\lambda\,(1 + \mu) = v \tag{23}$$

where v is the row vector of values and $\mu$ is the rate of surplus value. Eliminate exploitation ($\mu = 0$) and the two equations, (21) with $\pi = 0$ and (23) with $\mu = 0$, become identical.

Since $p^* = v^*$, the substitution in the two equations gives

$$(p^*\,A + w\,\lambda)\,\pi = w\,\lambda\,\mu \tag{24}$$

Since $p^*A = \omega\,(w\lambda)$, after another substitution we obtain

$$\mu = (\omega + 1)\,\pi \tag{25}$$

which says the rate of surplus value (the rate of exploitation) is proportional to the rate of profit, the organic composition of capital plus one representing the proportionality factor. This is the Marxist version of the Sraffian linear relation as can be verified by using the definitions of the variables.

Marx imposed two conditions on the solution of the transformation problem (1) that the sum of prices of production be equal to the sum of values and (2) that the sum of profits be equal to the sum of surplus values. Since Bortkeinicz,[4] the two conditions have been considered contradictory in a general case. And so they are *if* the organic composition generally cannot be made uniform. Bortkiewicz and his followers worked with the variable capital *V* instead of with $V = w\lambda$ and so they overlooked an additional degree of freedom inherent in the system. If this degree of freedom is utilized, the organic composition can be made uniform. Consequently, Marx's problem has always a solution. The proof is trivial. Since $p^* = v^*$, the first condition is fulfilled. The second condition is satisfield by (24). The two conditions are satisfied not only in the case of simple reproduction, but also under extended reproduction characterized by uniform growth.

An interesting reinterpretation of Marx is now possible. Marx assumed that in stagnant precapitalist societies producers were *not* motivated by profits but insisted on equal remuneration of work. Thus, relative prices coincided with relative labour values and were determined by the following equation.

$$pA + W\lambda = p = v \tag{26}$$

When capitalists appeared on the market, the *same* competitive prices still obtained with profit, wages and surplus value entering the price formation.

In other words, the transformation of values into prices changes only ownership relations and the consequential distribution of income. Technology, productivity and income remain unchanged.

$$(pA + w\lambda)(1 + \pi) = p = pA + w(1 + \mu)\lambda \tag{27}$$

The resulting surplus value appears as transformed into profit as described by (24). But (27) does not guarantee that $\pi$, w and $\mu$ are uniform throughout the economy. And we cannot just fix them arbitrarily because some prices may turn to be negative. However, if medieval artisans followed the same pricing rule as modern artisans and added a margin proportional to their labour cost in order to cover the cost of material inputs *and* the proportionality rate was uniform across the trades, i.e.

[4] L. von Bortkiewicz, »On the Correction of Marx's Fundamental Theoretical Construction in the Third Volume of *Capital*« originally published in *Jahrbücher für Nationalökonomie und Statistik*, July, 1907. For a history of the transformation problem see B. Horvat (1987).

$$pA = \omega (W\lambda)$$

we know from (22) that the resulting prices will do the trick. Note that technology, as described by A and $\lambda$ is left undisturbed. All we need is a simple pricing rule. The transformation of the rate of surplus value into the rate of profit is given by (25).

I do not know whether craftsmen in medieval towns used the uniform $\omega$ rule when setting prices for their customers. Yet such an assumption does not seem more far-fetched — and is possibly much less so — than the fundamental postulate of the neoclassical distribution theory which requires that the world be linearly homogeneous.

## CONCLUSION

In the same way that Ricardo was searching for an invariable standard of value, Marx was searching for a transformation procedure which would leave net output (profits, surplus value) and total output of the economy invariant whether expressed in prices of production or in values. Sraffa's standard commodity solves Ricardo's problem in the sense that income distribution leaves the value of the constructed composite commodity invariant. Marx's problem is solved by imposing a uniform organic composition of capital on the economy without changing technology. The solutions are not ideal: standard commodity is not fully invariant nor is the transformation procedure fully general. But they are helpful. The solutions of both problems turn out to be identical and reduce to the application of Perron-Frobenius theorems concerning non-negative indecomposable matrices which neither Ricardo nor Marx knew since they had been long dead before the theorems were discovered. Economics seems to be becoming a science, after all.

*Received: 15. 01. 1987*
*Revized: 20. 09. 1987*

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