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RATE OF CHANGE IN ECONOMIC RESEARCH*

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1. INTRODUCTION

Certain economic phenomena are often considered in research through rate of change. Rates of change represent information which enable researchers to come to conclusions about tendencies and the intensities of observed phenomena.

In applications the computation of rate of change is mostly carried out by using the geometric mean of ratios of data (e. g. see [8]). The basic disadvantage of such an approach is the elimination of the influence of all data (except two of them), because the conclusion is carried out on the basis of the first and last datum. Besides, using the geometric mean, the rate of change can be computed only for the equidistant distributed data, and this also represents a limitation in its application.

This paper points out the disadvantages of such an approach, introduces precise definition of the rate of change and proposes the manner in which this important economic inductor can be more accurately determined.

2. THE RATE OF CHANGE

Let variable y depends on some dependent variable x as $y = f(x)$, where $f: D \rightarrow R$ $(D \subseteq R)$ is a derivable function on D.

DEFINITION 1. *Relative ratio of change in the value of the dependent variable y in an interval Δx*

$$c = \frac{1}{y} \frac{\Delta y}{\Delta x} \tag{1}$$

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will be called the rate of change of the dependent variable y in the interval Δx. If $c > 0$, or $c < 0$ we speak about the rate of growth, or the rate of decrease, respectively.

REMARK 1. *In economic research the independent variable x often represents time. If the interval Δx is a year, then $c = \Delta y/y$ is termed the yearly rate of change of the dependent variable y which is in most cases expressed in percentages.*

If in (1) it is allowed that $\Delta x \to 0$, then the following definition can be introduced:

DEFINITION 2. *Let $D \subseteq R$ and $f : D \to R$ be a derivable function in the point $x_0 \in D$. If $f(x_0) \neq 0$, then the number*

$$c = \frac{1}{f(x_0)} f'(x_0) \tag{2}$$

will be called the rate of change of the function $y = f(x)$ in the point x_0.

3. DETERMINATION OF THE RATE OF GROWTH OR THE RATE OF DECREASE

In economic research mapping f (according to this the dependent variable y depends on the independent variable x) is most often not known. This mapping can be determined by knowing the nature of the dependent variable y, and on the basis of a certain number of data (p_i, x_i, y_i), $i = 1, \ldots, m$ which characterize the dependent variable y in points $x_i \cdot p_i$ are the positive numbers representing weights of data. Generally, the function f has the form

$$x \to f(x; a_1, \ldots, a_n), \qquad n < m,$$

where $a_1, \ldots, a_n$ are parameters which can be determined e. g. by the method of least squares (see [3], [4], [5]), i. e. by minimization of the functional

$$F(a_1, \ldots, a_n) = \sum_{i=1}^{m} p_i \left[f(x_i; a_1, \ldots, a_n) - y_i \right]^2 \tag{3}$$

If the functions $a_j \to f(a., \ldots, a_j \ldots)$ are linear for each $j = 1, \ldots, n$, then there is a linear problem of least squares (see [2], [6]). Opposed to this there is a nonlinear problem of the least squares (see [3], [4], [5]).

The nonlinear least squares problem is a special case of unconstrained minimization, which is generally solved using iterative methods of the form

$$a_{k+1} = a_k + \lambda_k p_k, \qquad k = 0, 1, \ldots \tag{4}$$

where p_k is the direction vector from the point $a_k = (a_1^{(k)}, \ldots, a_n^{(k)})^T$ to the point a_{k+1}, and λ_k is the length of the step in the direction of the the vector p_k. At the same time the minimizing function (3) has a special form. Its gradient can be written in the form

$$grad\, F = J^T P \Phi,$$

where $\Phi : R^n \to R^m$ is a function with components $\Phi^i(a) = y_i - f(x_i; a)$, J — Jacobian matrix of the function:

$$J = \begin{bmatrix} \frac{\partial \Phi^1}{\partial a_1} & \cdots & \frac{\partial \Phi^1}{\partial a_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial \Phi^m}{\partial a_1} & \cdots & \frac{\partial \Phi^m}{\partial a_n} \end{bmatrix},$$

and $P = diag\,(p_1, \ldots, p_m)$. Hessian of the functional F can be written as

$$H_F = J^T P J + \sum_{k=1}^{m} p_k \Phi^k H_k,$$

where $H_k \in L(R^n)$ are the matrices defined with

$$(H_k)_{ij} = \frac{\partial^2 \Phi^k}{\partial a_i\, \partial a_j}, \qquad k = 1, \ldots, m.$$

Most algorithms for solving nonlinear least squares problems use these special forms of the gradient and the Hessian of the functional F.

If in (4) vector p_k has been chosen as the solution of equation

$$J_k^T P J_k p = -grad\, F_k$$

then we get the well-known Gauss-Newton method (see [3], [5]), which is used for the requirements of this paper. Some other methods for solving nonlinear least squares problems can be also seen in [3] and [5].

This paper shows how the average rate of change (growth or decrease) of the dependent variable y can be determined in a certain interval taking into account known values of the dependent variable y in this interval and using the least squares method.

Suppose that data (p_i, x_i, y_i), $i = 1, \ldots, m$ are known, where $x_1 < x_2 < \ldots < x_m$. The function $x \to f(x)$ should be determined so that in each point of the interval $[x_1, x_m]$ there is the constant rate of change, and at the same time the values of function f in points $(x_i, i = 1, \ldots, m)$ should be as close as possible to the values $(y_i, i = 1, \ldots, m)$. Coefficients k and l of the associated linear trend $y = kx + l$ in the sense of the least squares method (see [6]) can always be uniformly defined for the given data.

DEFINITION 3. *We will say that data* (p_i, x_i, y_i), $i = 1, \ldots, m$ *have the property of the preponderant decrease (growth) if the coefficient of direction k of associated linear trend is negative (positive).*

For a more detailed explanation of this notion see [7].

In this work the following theorem is proved:

THEOREM 1. *Let real numbers* $x_1 < x_2 < \ldots < x_m, y_1, \ldots, y_m$ *as well as positive numbers* $p_1, \ldots, p_m$ *be given and let*

(i) $y_i > 0$, $i = 1, \ldots, m$

(ii) *data* (p_i, x_i, y_i), $i = 1, \ldots, m$ *have the property of preponderant decrease.*

In such case there are numbers $b^* > 0$ *and* $c^* < 0$ *by which the functional*

$$F(b, c) = \sum_{i=1}^{m} p_i [b \exp(cx_i) - y_i]^2 \tag{5}$$

is minimized on the set $A = \{(b, c) \in R^2 \; b > 0, c < 0\}$.

By analogy, it could also be shown in the case when data have the property of the preponderant growth. Namely, it can be shown that in such a case there are numbers $b^* > 0$ and $c^* > 0$ which minimize the functional (5).

THEOREM 2. *Let* (p_i, x_i, y_i), $i = 1, \ldots, m$ *be given data and* $x_1 < \ldots < x_m$. *Function* $x \to f(x; b^*, c^*) = b^* . \exp(c^*, x)$, *where* (b^*, c^*) *is the minima point of the functional (5) has a constant rate of growth in each point* $x \in R$ *(if data have the property of the preponderant growth), or decrease (if data have the property of the preponderant decrease) which equals to* c^*. *At the same time the sum of squared deviations* $(f(x_i) - y_i)^2$ *related to weights* p_i *is the least.*

Proof. Let it be assumed that data (p_i, x_i, y_i), $i = 1, \ldots, m$ have the property of the preponderant decrease. The function with the property of such rate of decrease which is constant and equal to c in each point $x \in R$ must satisfy the *Definition* 2, i. e.

$$\frac{1}{y} y' = c, \quad c < 0. \tag{6}$$

Solution of this differential equation is the function

$$f(x; b, c) = b \exp(cx). \tag{7}$$

For the given set of data (p_i, x_i, y_i), $i = 1, \ldots, m$ which has the property of preponderant decrease according to Theorem 1, there is always a

pair of numbers (b^*, c^*) $(b^* > 0, c^* < 0)$ by which the functional (5) is minimized.

A similar approach can be used to show that in the case of preponderant growth of data there is always a pair of numbers (b^*, c^*) $(b^* > 0, c^* > 0)$ by which the functional (5) is minimized.

The problem of the determination of parameters b and c in exponential function $x \to b \exp(cx)$ using the method of least squares can be transformed in linear form so that instead of minimization of functional (5) it should be minimized as the functional

$$\Phi(b, c) = \sum_{i=1}^{m} p_i [\ln b + cx_i - \ln y_i]^2 \tag{8}$$

which is well known in statistical literature (see, e. g. [8], [9]). In such a case the required condition is that the sum of squared logarithmic deviations of real values (y_i) from theoretical ones $(f(x_i))$ should be minimal.

When x is time and if the rate of changes should be approximated in the interval $[0, T]$, then instead of (7) the function (see [9])

$$f(t; c) = f(0)(1 + c)^t \tag{9}$$

is used more often for the small values of c. If $t = T$ and using notation $y_0 = f(0)$ and $y_T = f(t)$, then (9) becomes

$$c = \sqrt[T]{\frac{y_T}{y_0}} - 1. \tag{10}$$

The first term in (10) can be considered as the geometric mean of ratios

$$\frac{y_1}{y_0}, \frac{y_2}{y_1}, \ldots, \frac{y_T}{y_{T-1}}.$$

It is obvious that using these formulas the influence of all other data except y_0 and y_T is completely lost and this always produces the incorrect value of rate of change c. In some cases the use of this approach can produce a completely wrong conclusion about the direction of the tendency in the observed phenomenon.

4. COMPARISON OF METHODS FOR COMPUTING THE RATE OF CHANGE

The rate of change of some observed economic phenomenon which is manifested through data (p_i, x_i, y_i), $i = 1, \ldots, m$ can be computed using formula (10) or by minimizing the functional (5), or (8).

Too often the rate of change is computed using the formula (10), but at the same time shortages observed in application of this method are not taken into the account. These shortages are expressed in the following:

— the computed rate of change is a direct result of the first and last datum, so

— all other data, between the first and last one, have no influence on the value of the rate of change.

If the rate of change is computed on the basis of minimizing the functional (5), the precise rate of change is obtained in the observed interval. The computational work is greatly simplified using linearization of (8), which leads to the frequent applications of this method. On the other hand, reasons for such simplification in measuring the rate of change are less justified especially if the importance of this economic indicator is taken into account as well as the presence of the fast development of methods for solving nonlinear least squares problems and the rapid dispersion of using a computer in computational work.

According to Theorem 1 there are always optimal parameters b^* and c^* of the exponential function $f(x) = b \exp(cx)$ by which the functional (5) is minimized. Therefore, if the minimum of the functional (8) is achieved in the point $(\hat{b}, \hat{c})$, it holds

$$F(b^*, c^*) \leq F(\hat{b}, \hat{c}). \qquad (11)$$

A simple and frequently occurring situation in which strict inequality exists in (11) will be shown in the following example.

EXAMPLE

Data $(1, -T, f_1)$, $(1, 0, f_2)$, $(1, T, f_3)$ are given, where positive numbers do not construct a geometric series and $f_1 \neq f_3$.

Values of optimal parameters obtained by minimizing functional (8) are

$$\hat{b} = \sqrt[3]{f_1 f_2 f_3}$$

$$\hat{c} = \frac{1}{2T} \ln \frac{f_3}{f_1} \qquad (12)$$

We will show that the functional (5) in the point $(\hat{b}, \hat{c})$ does not achieve its minimum which according to *Theorem 1* means that

$$F(\hat{b}, \hat{c}) > F(b^*, c^*).$$

The second component of the gradient of the functional F can be seen in the point $(\hat{b}, \hat{c})$:

$$A = \frac{1}{2} \frac{\partial F(b, c)}{\partial c} = \hat{b} T (-\hat{b} e^{-2\hat{c}T} + f_1 e^{-\hat{c}T} + \hat{b} e^{2\hat{c}T} - f_3 e^{\hat{c}T})$$

which can be written in the form

$$A = \hat{b} T \frac{(f_3^2 - f_1^2)(\sqrt[3]{f_1 f_2 f_3} - \sqrt{f_1 f_3})}{f_1 f_3} \qquad (13)$$

The expression $f_3^2 - f_1^2$ is not cancelled because of the condition $f_1 \neq f_3$, and the expression $\sqrt[3]{f_1 f_3 f_2} - \sqrt{f_1 f_3}$ also does not disappear, since we assumed that numbers f_1, f_2, f_3, do not construct a geometric series. It means that the point (b, c) is not a stationary point of the functional (5), and neither can its minimum be achieved in it.

REMARK 2. This example shows that the rates of change computed on the basis of the functionals (5) and (8) (in the case when $n = 3$ and equidistant distributed x-axis) are the same only if numbers f_1, f_2, f_3 are members of the geometric series.

It is expected that this result can be generalized.

For example, if data are given as

$(1, -1, 0.01)$, $(1, 0, 0.05)$; $(1, 1, 0.9)$

using the formula (10) we get $s' = 8.49$.

Since

$$\Delta y = b\, e^{cx} (e^c - 1)$$

holds for the exponential function (7) in the case when $\Delta x = 1$, it can be easily shown that according to *Definition 1* the rate of change is given

$$s = e^c - 1.$$

Therefore, by minimizing the functional (8) value of the rate of change $\hat{s} = 8.488$, but the real value of the rate of change computed on the basis of minimizing the functional (5) is $s = 16.64$.

5. CONCLUSION

In the case when formula (10) is used for computing the rate of change of the dependent variable y, the interpretation of the result obtained should take into account all the limitations of this method. These limitations arise from the fact that the exponential function is linearized using Taylor's formula and can therefore only be used in cases of a distinct stable exponential growth of the dependent variable y. In estimation of parameters b and c in exponential function by minimizing functional (8) there is supposed the existence of strict positiveness of data. If data y_i are close to 0, there is no reliability in the parameters and the rate of growth estimated in this manner.

On the basis of all the results considered, it can be concluded that the rate of change can be estimated with the greatest reliability using the method of minimizing the functional (5), and this means that some of the methods for solving nonlinear least squares problems should be used.

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STOPA PROMJENE U EKONOMSKIM ISTRAŽIVANJIMA

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Rezime

U svrhu praćenja tendencije i intenziteta neke ekonomske kategorije često je nužno sa sigurnošću poznavati kretanje njene stope rasta ili pada.

U ovom članku uvodi se precizna definicija stope rasta ili pada promatrane ekonomske pojave, te predlaže način kako se egzaktno može procijeniti ovaj važan ekonomski pokazatelj.

Naime, u slučaju kada se za izračunavanje stope promjene zavisne varijable y koristi formula (10), tada se u interpretaciji dobivenog rezultata treba voditi računa o svim ograničenjima ovakvog pristupa. Ograničenja nastaju iz činjenice da se primjenom ove metode eksponencijalna funkcija linearizira korištenjem Taylorove formule, te se zbog toga može koristiti samo u slučajevima izrazito stabilnog eksponencijalnog rasta zavisne varijable y. U procjeni parametara b i c u eksponencijalnoj funkciji minimiziranjem funkcionala (8) pretpostavlja se egzistencija stroge pozitivnosti podataka. Međutim, ako su podaci y_i *blizu 0, tada parametri i stopa promjene procjenjeni na ovaj način nisu vjerodostojni.*

Na osnovi svih razmatranih rezultata može se zaključiti da je stopa promjene procjenjena korištenjem metode minimiziranja funkcionala (5) najvjerodostojnija, ali da takav pristup zahtijeva primjenu nekih metoda rješavanja nelinearnih problema najmanjih kvadrata.